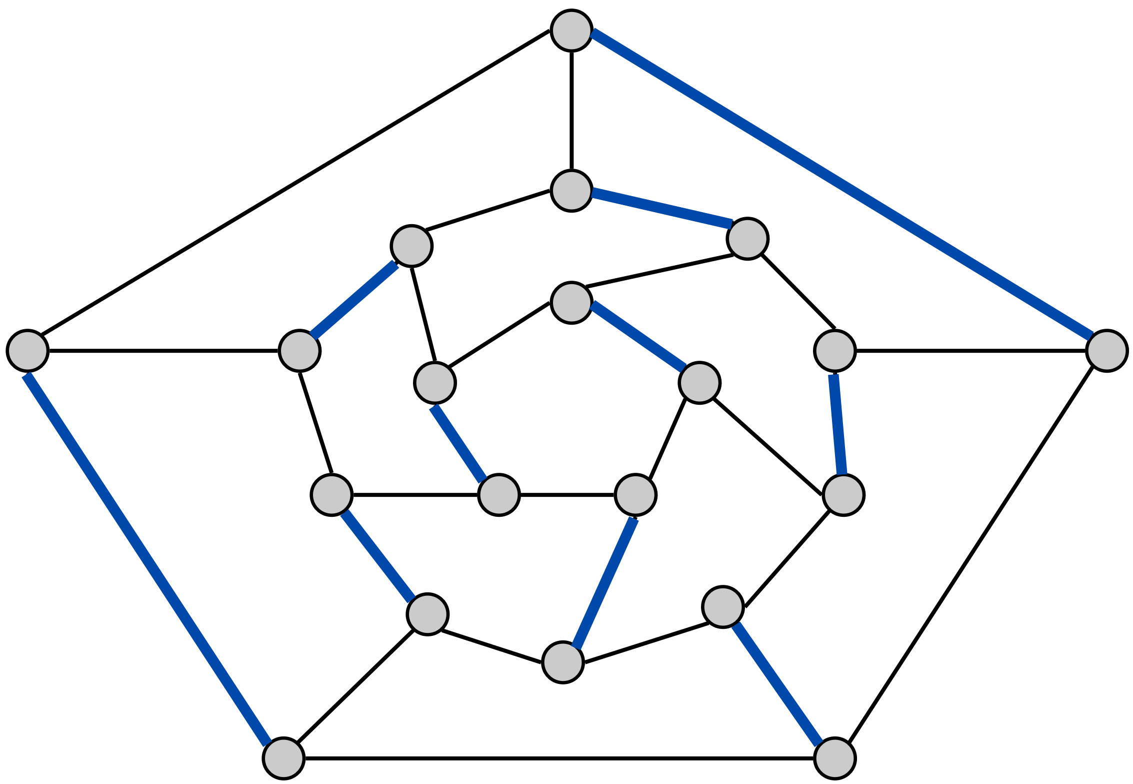
7.5 Bipartite Matching

Matching

Matching.

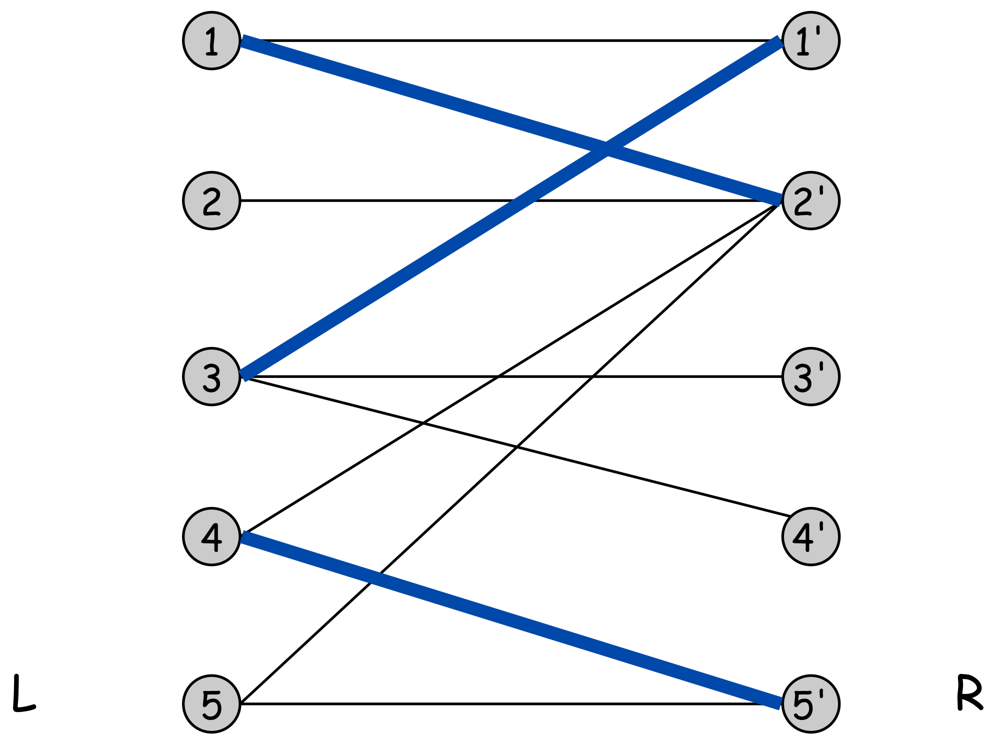
- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in M.
- Max matching: find a max cardinality matching.



Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in M.
- Max matching: find a max cardinality matching.



matching

1-2', 3-1', 4-5'

Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in M.
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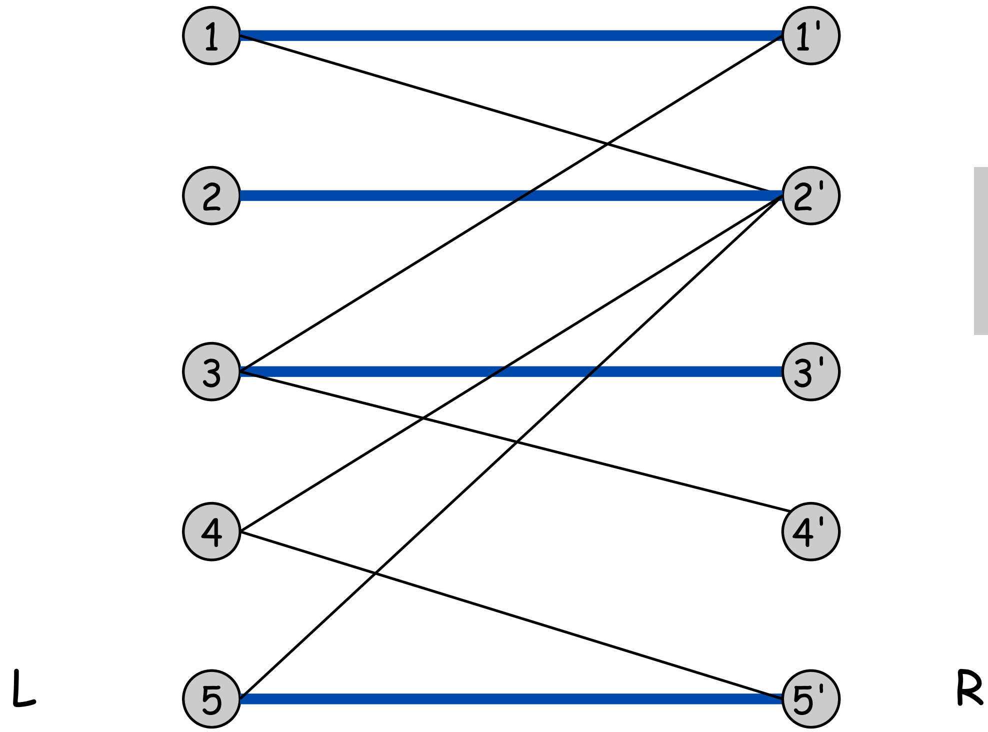


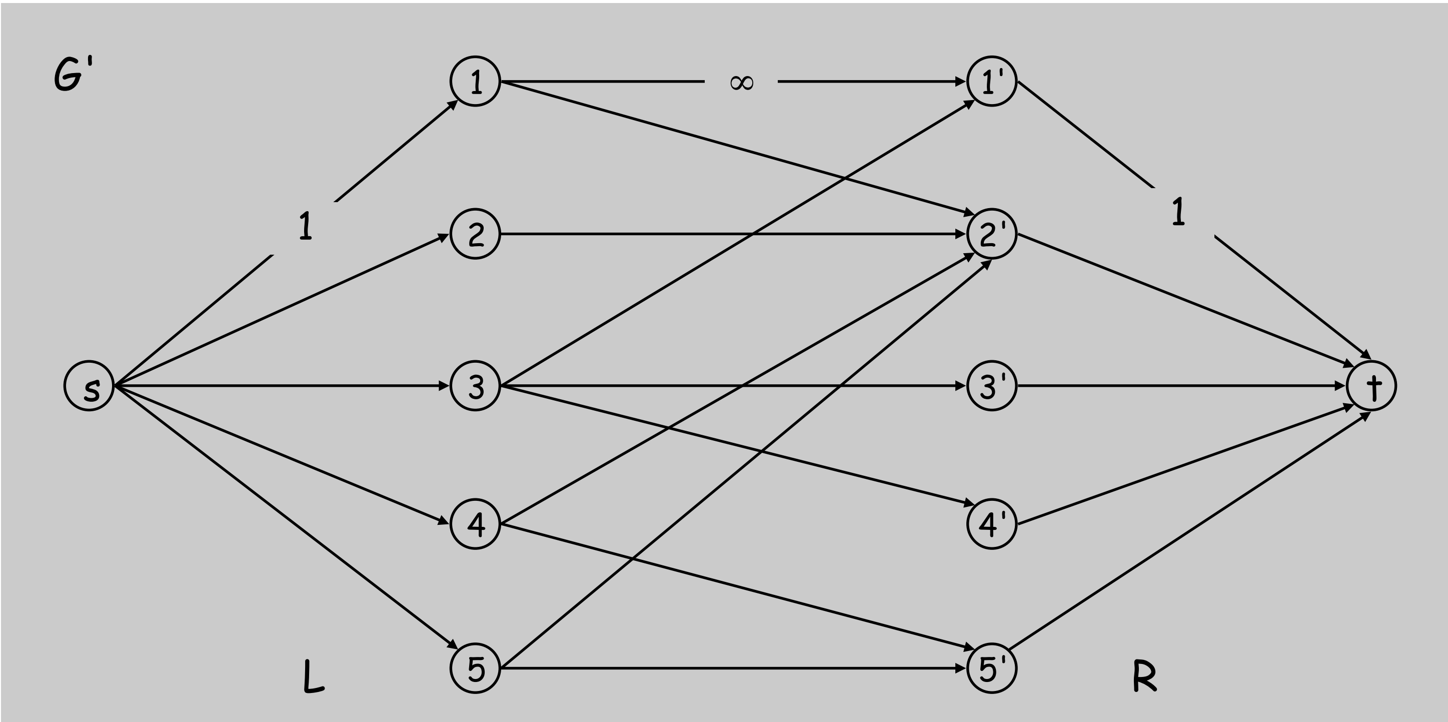
max matching

1-1', 2-2', 3-3' 4-4'

Bipartite Matching

Max flow formulation.

- Create digraph $G' = (L \cup R \cup \{s, t\}, \ E')$.
- Direct all edges from L to R, and assign infinite (or unit) capacity.
- Add source s, and unit capacity edges from s to each node in L.
- Add sink t, and unit capacity edges from each node in R to t.

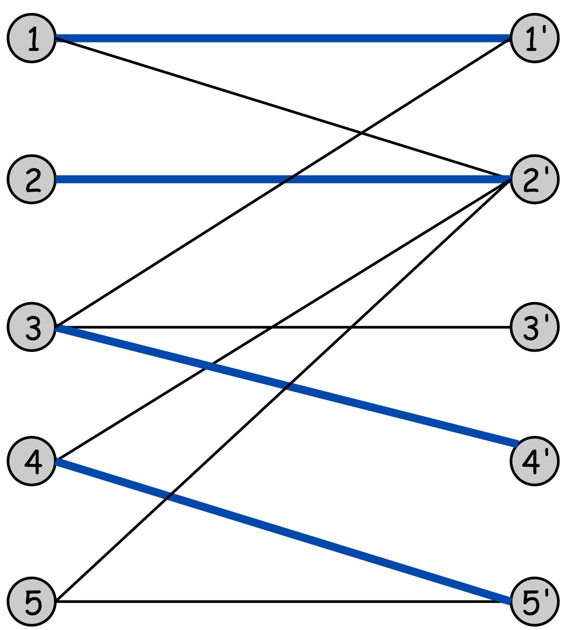


Bipartite Matching: Proof of Correctness

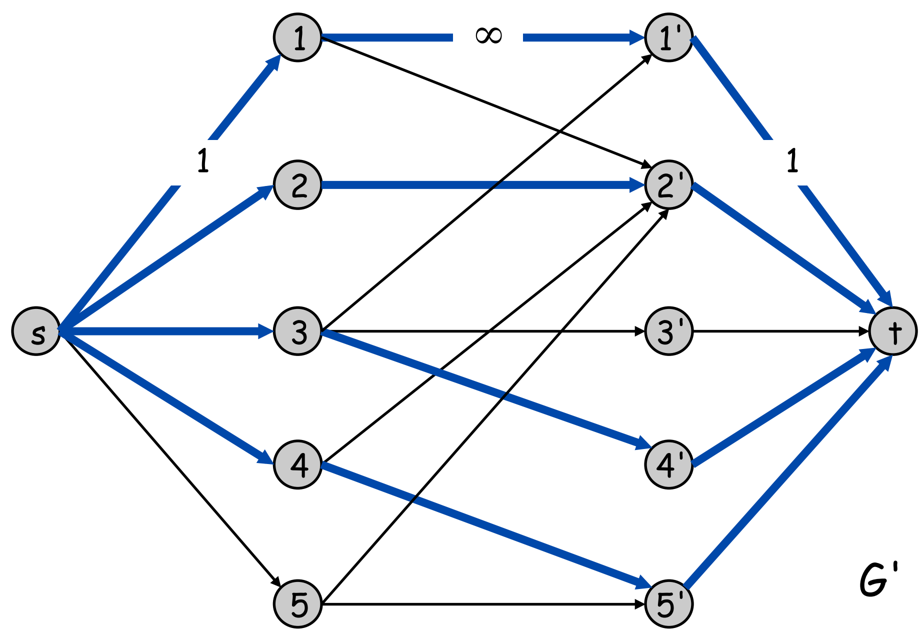
Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. $\leq$

- Given max matching M of cardinality k.
- Consider flow f that sends 1 unit along each of k paths.
- f is a flow, and has cardinality k. ▪



G



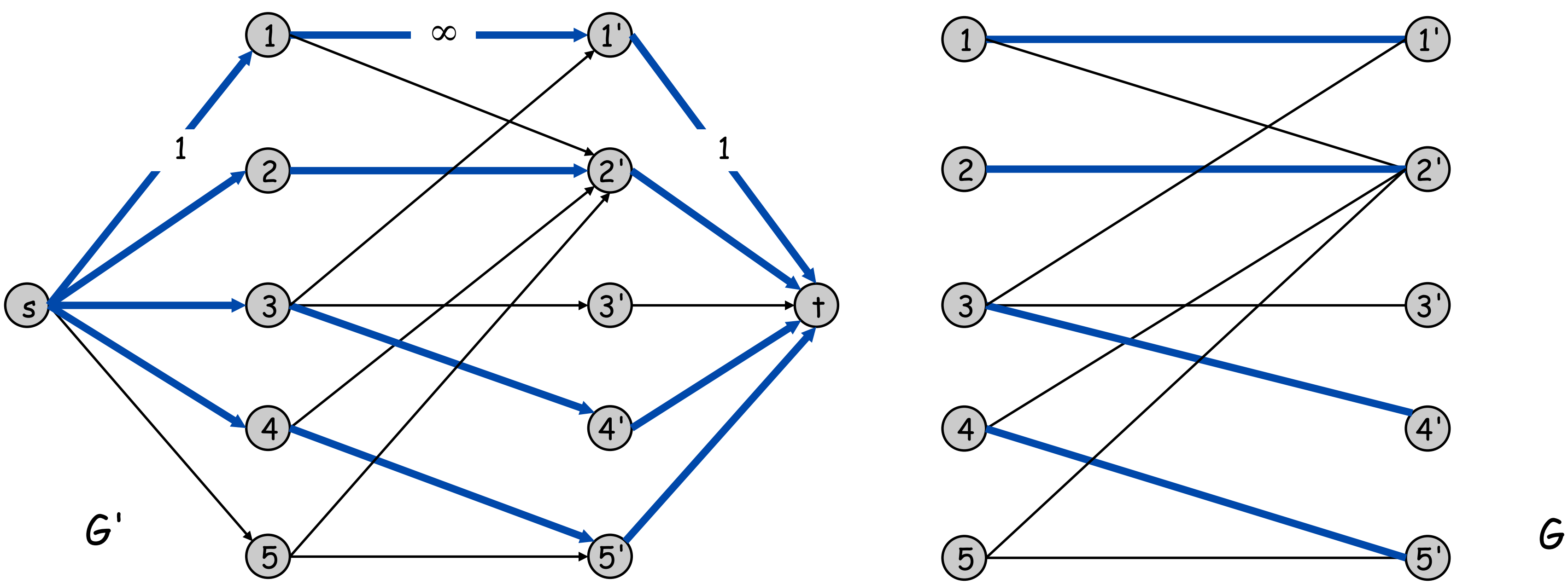
G'

Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. $\geq$

- Let f be a max flow in G' of value k.
- Integrality theorem $\Rightarrow$ k is integral and can assume f is 0-1.
- Consider M = set of edges from L to R with f(e) = 1.
  - each node in L and R participates in at most one edge in M
  - $|M|$ = k: consider cut ($L \cup s$, $R \cup t$) ▪



Perfect Matching

Def. A matching $M \subseteq E$ is perfect if each node appears in exactly one edge in M.

Q. When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings.

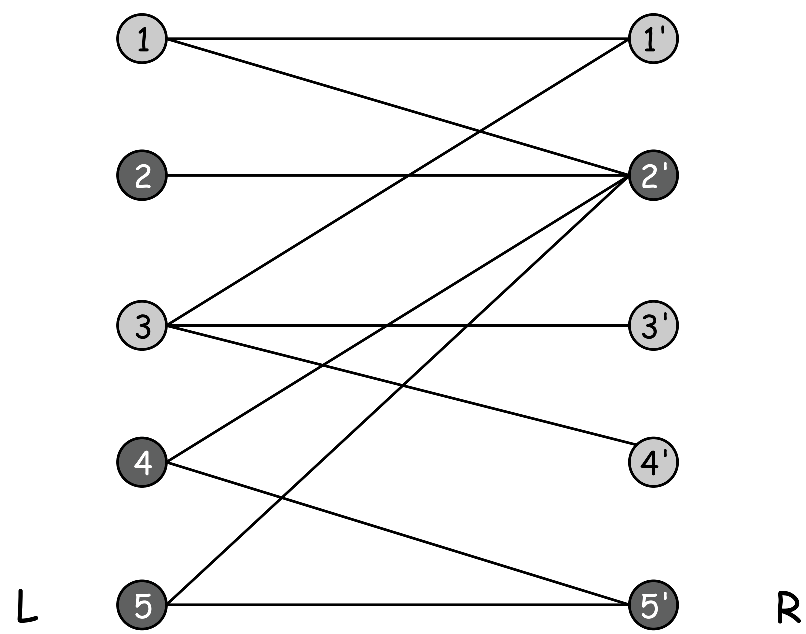
- Clearly we must have $|L| = |R|$.
- What other conditions are necessary?
- What conditions are sufficient?

Perfect Matching

Notation. Let S be a subset of nodes, and let N(S) be the set of nodes adjacent to nodes in S.

Observation. If a bipartite graph $G = (L \cup R, E)$, has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. Each node in S has to be matched to a different node in N(S).



No perfect matching:

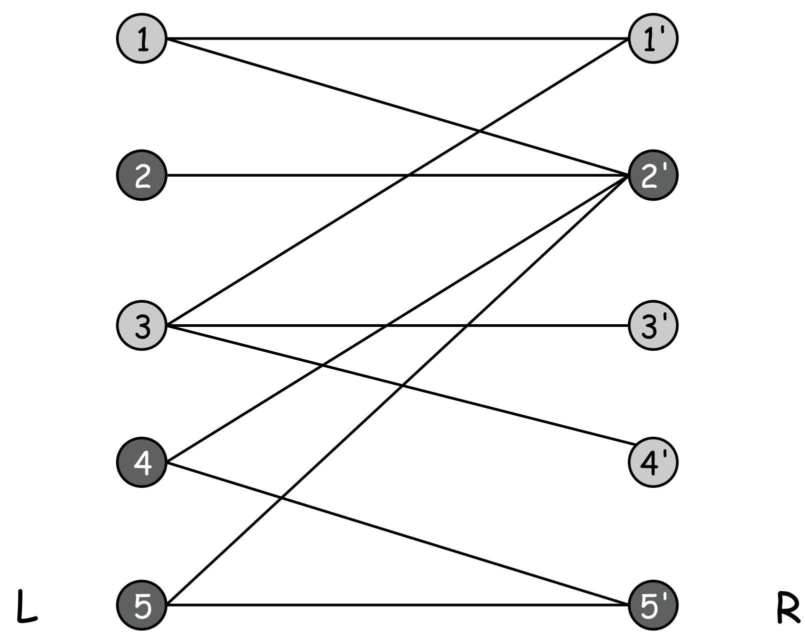
S = { 2, 4, 5 }

N(S) = { 2', 5' }.

Marriage Theorem

Marriage Theorem. [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then, G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. $\Rightarrow$ This was the previous observation.



No perfect matching:

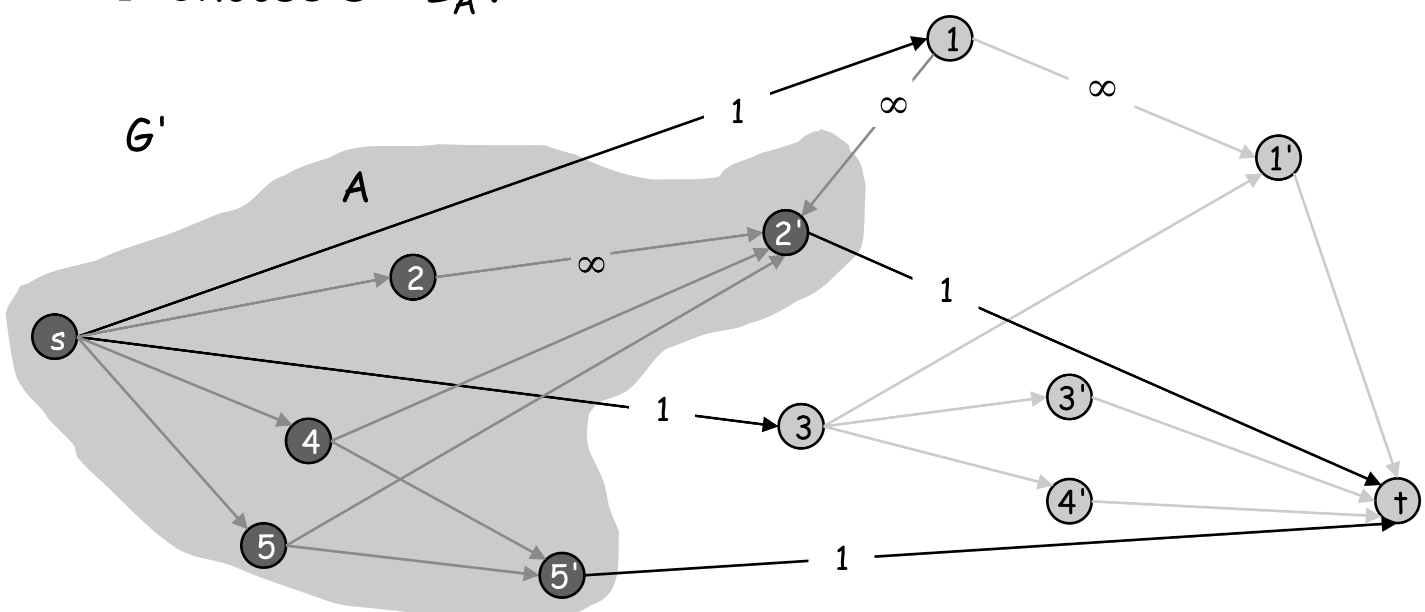
$S = \{ 2, 4, 5 \}$

$N(S) = \{ 2', 5' \}$.

Proof of Marriage Theorem

Pf. $\Leftarrow$ Suppose G does not have a perfect matching.

- Formulate as a max flow problem and let (A, B) be min cut in G'.
- By max-flow min-cut, cap(A, B) < | L |.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
- $cap(A, B) = |L_B| + |R_A|$.
- Since min cut can't use ∞ edges: $N(L_A) \subseteq R_A$.
- $|N(L_A)| \le |R_A| = cap(A, B) - |L_B| < |L| - |L_B| = |L_A|$.
- Choose $S = L_A$. ▪



$L_A = \{2, 4, 5\}$
$L_B = \{1, 3\}$
$R_A = \{2', 5'\}$
$N(L_A) = \{2', 5'\}$

Bipartite Matching: Running Time

Which max flow algorithm to use for bipartite matching?

- Generic augmenting path: $O(m \operatorname{val}(f^*)) = O(mn)$.
- Capacity scaling: $O(m^2 \log C) = O(m^2)$.
- Shortest augmenting path: $O(m n^{1/2})$.

Non-bipartite matching.

- Structure of non-bipartite graphs is more complicated, but well-understood. [Tutte-Berge, Edmonds-Galai]
- Blossom algorithm: $O(n^4)$. [Edmonds 1965]
- Best known: $O(m n^{1/2})$. [Micali-Vazirani 1980]

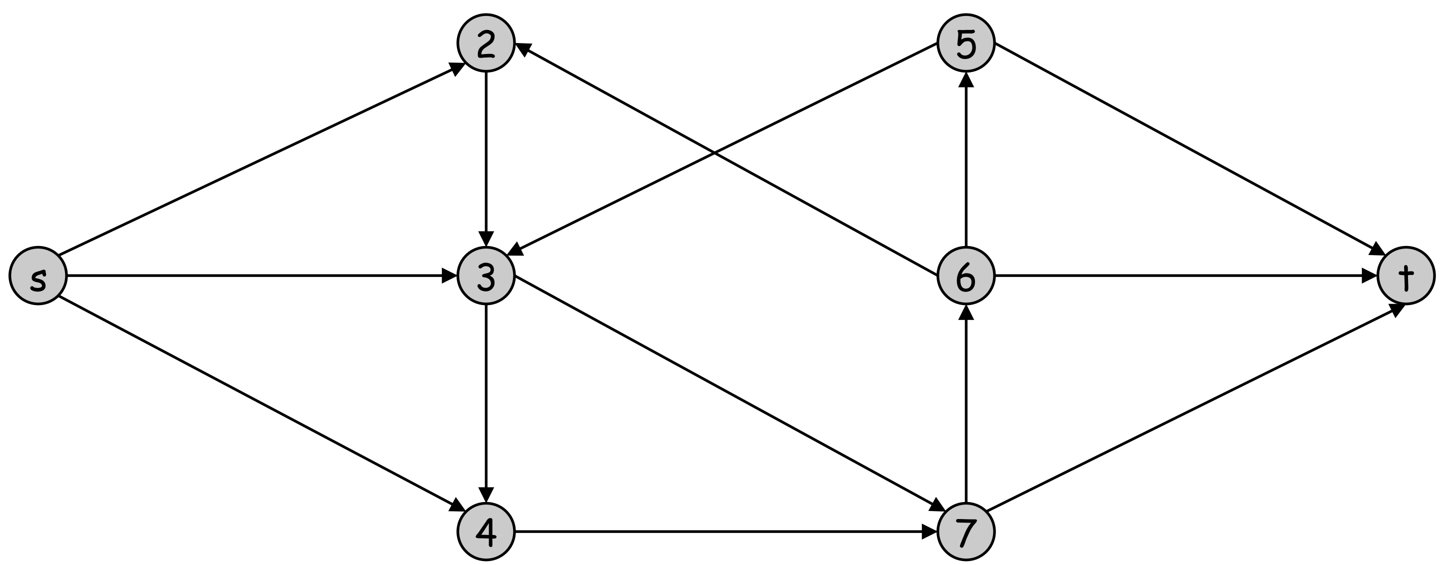
7.6 Disjoint Paths

Edge Disjoint Paths

Disjoint path problem. Given a digraph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths.

Def. Two paths are edge-disjoint if they have no edge in common.

Ex: communication networks.



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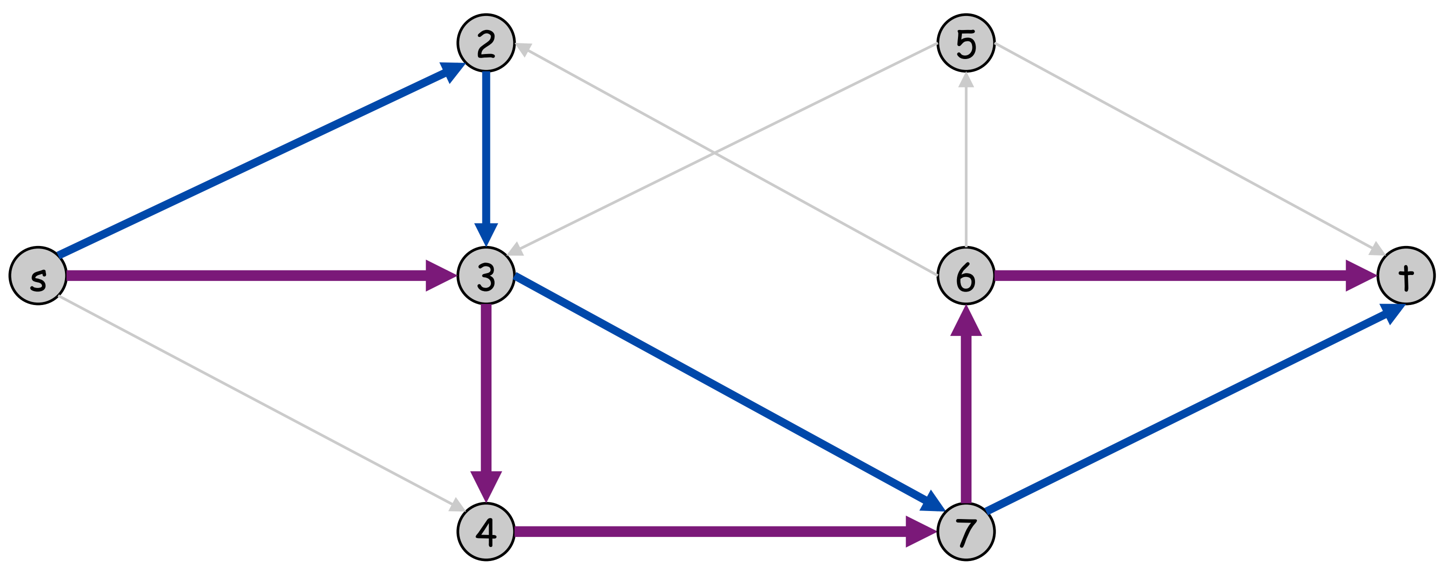
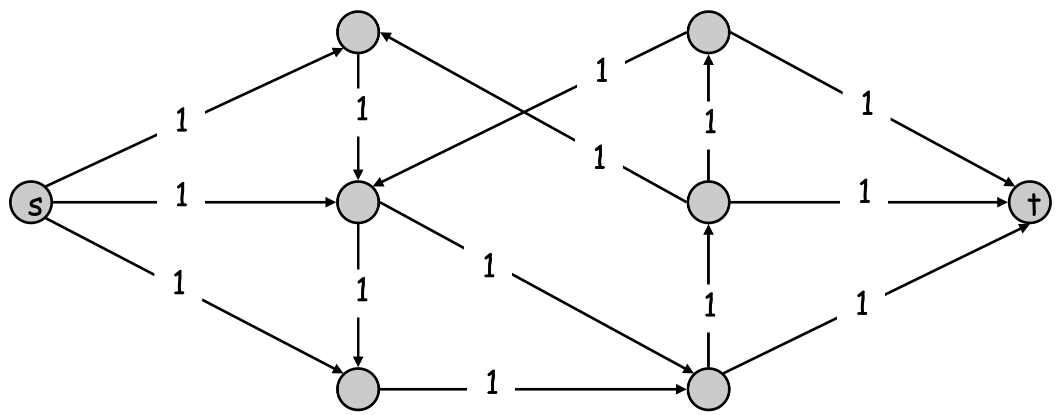


Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



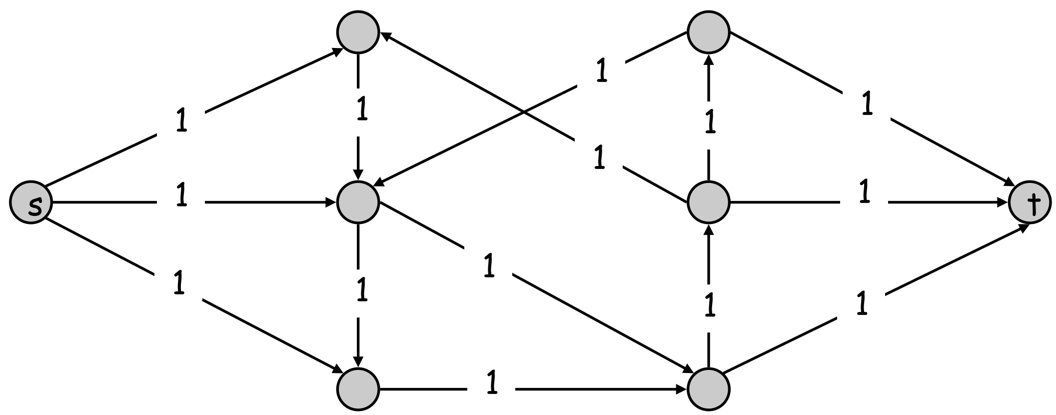
Theorem. Max number edge-disjoint s-t paths equals max flow value.

Pf. $\leq$

- Suppose there are k edge-disjoint paths $P_1, \ldots, P_k$.
- Set $f(e) = 1$ if e participates in some path P_i ; else set $f(e) = 0$.
- Since paths are edge-disjoint, f is a flow of value k. ▪

Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



Theorem. Max number edge-disjoint s-t paths equals max flow value.

Pf. $\geq$

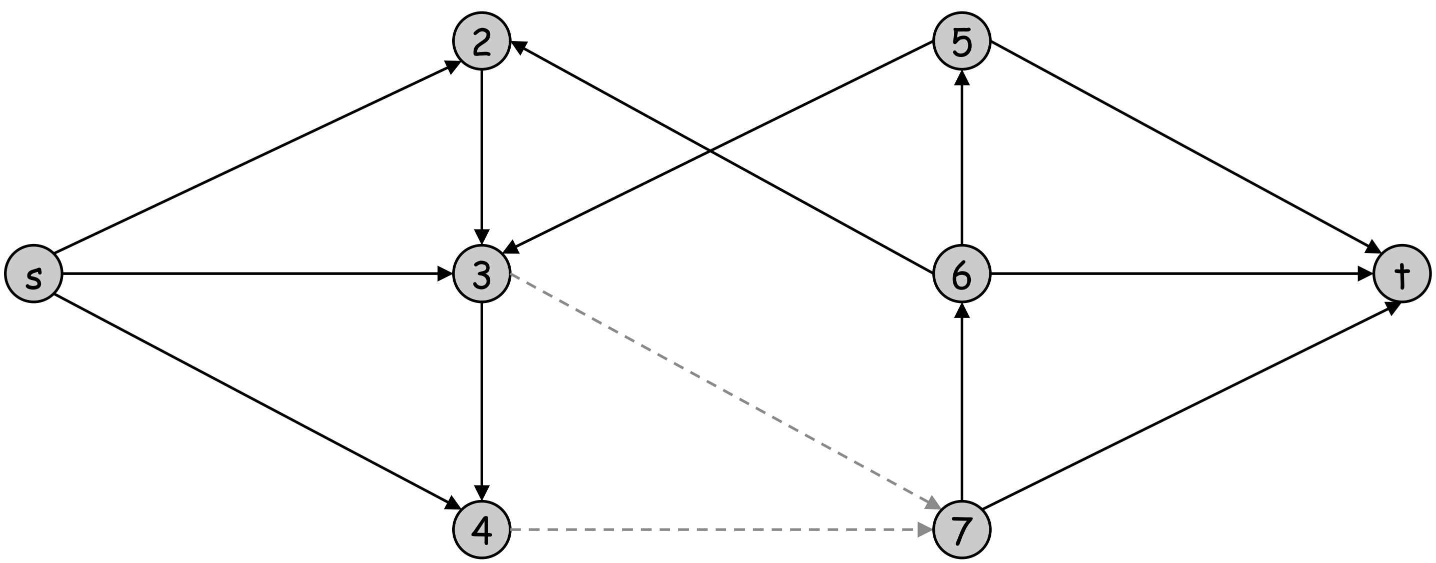
- Suppose max flow value is k.
- Integrality theorem $\Rightarrow$ there exists 0-1 flow f of value k.
- Consider edge (s, u) with f(s, u) = 1.
  - by conservation, there exists an edge (u, v) with f(u, v) = 1
  - continue until reach t, always choosing a new edge
- Produces k (not necessarily simple) edge-disjoint paths. ▪

can eliminate cycles to get simple paths if desired

Network Connectivity

Network connectivity. Given a digraph G = (V, E) and two nodes s and t, find min number of edges whose removal disconnects t from s.

Def. A set of edges $F \subseteq E$ disconnects t from s if all s-t paths uses at least on edge in F.

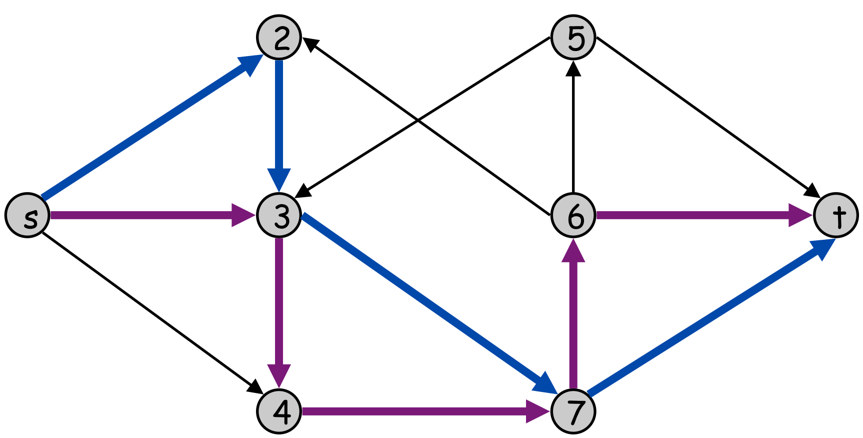


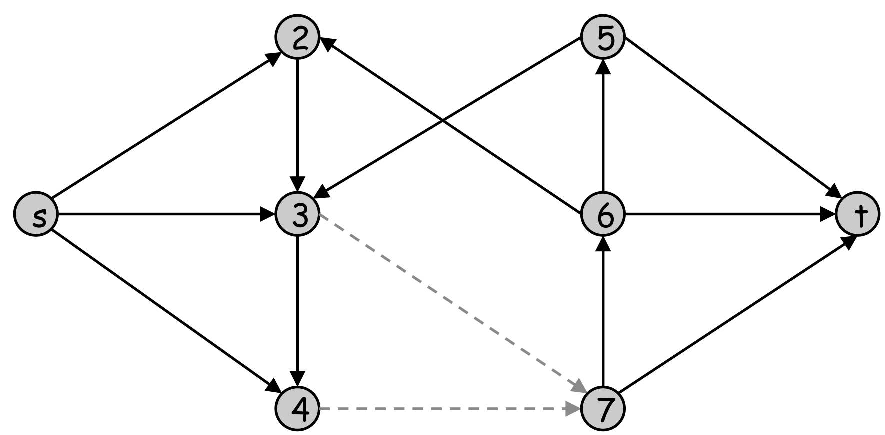
Edge Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf. $\leq$

- Suppose the removal of $F \subseteq E$ disconnects t from s, and $|F| = k$.
- All s-t paths use at least one edge of F. Hence, the number of edge-disjoint paths is at most k. ▪



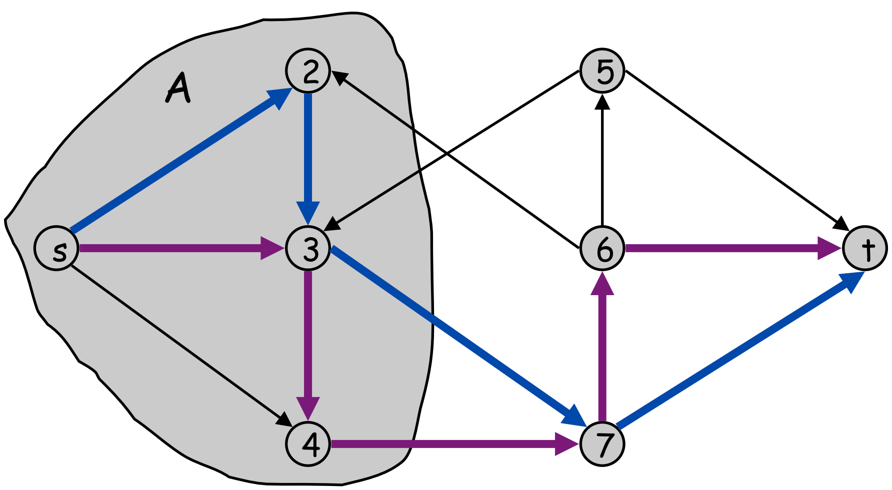


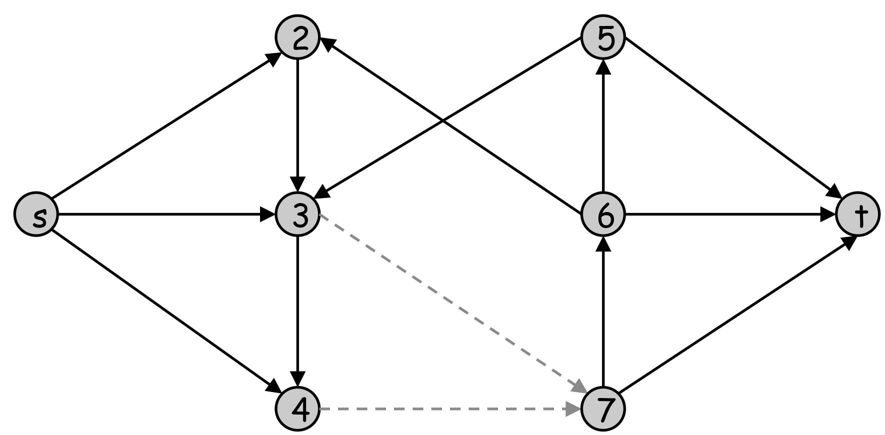
Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf. $\geq$

- Suppose max number of edge-disjoint paths is k.
- Then max flow value is k.
- Max-flow min-cut $\Rightarrow$ cut (A, B) of capacity k.
- Let F be set of edges going from A to B.
- |F| = k and disconnects t from s. ▪





7.7 Extensions to Max Flow

Circulation with Demands

Circulation with demands.

- Directed graph $G = (V, E)$.
- Edge capacities $c(e)$, $e \in E$.
- Node supply and demands $d(v)$, $v \in V$.

demand if $d(v) > 0$; supply if $d(v) < 0$; transshipment if $d(v) = 0$

Def. A circulation is a function that satisfies:

- For each $e \in E$: $\quad 0 \le f(e) \le c(e)$ (capacity)
- For each $v \in V$: $\quad \sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

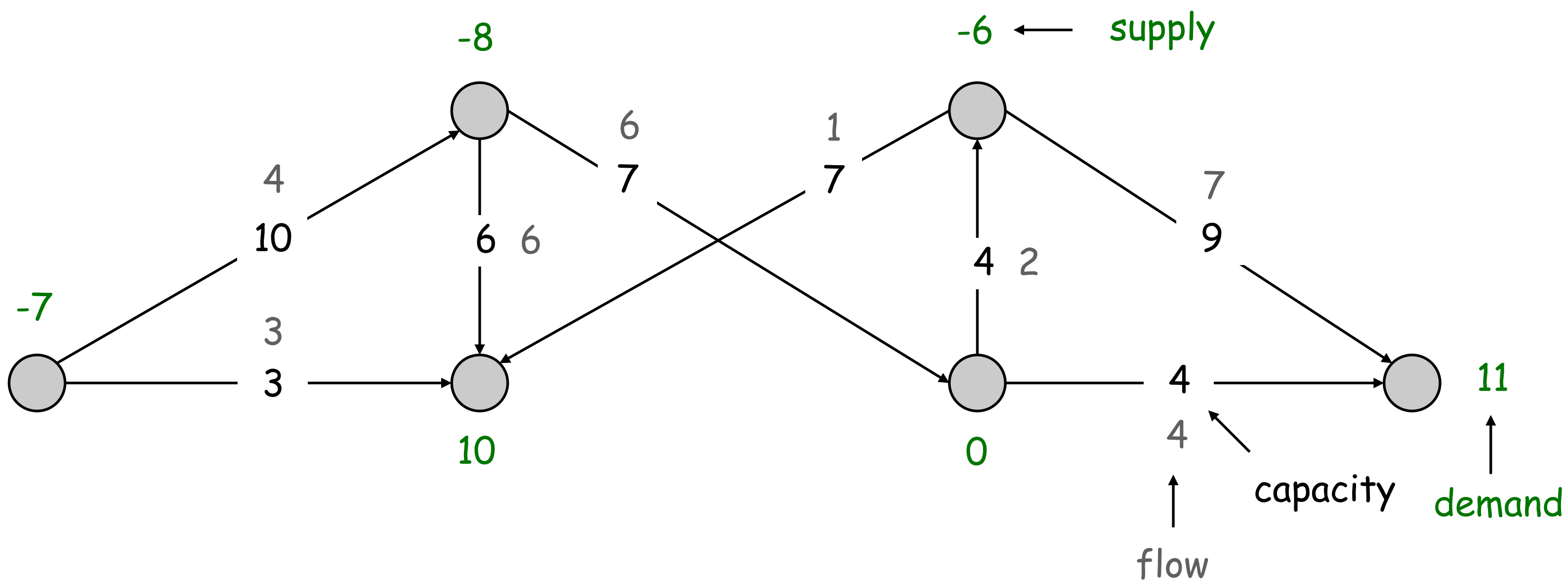
Circulation problem: given (V, E, c, d), does there exist a circulation?

Circulation with Demands

Necessary condition: sum of supplies = sum of demands.

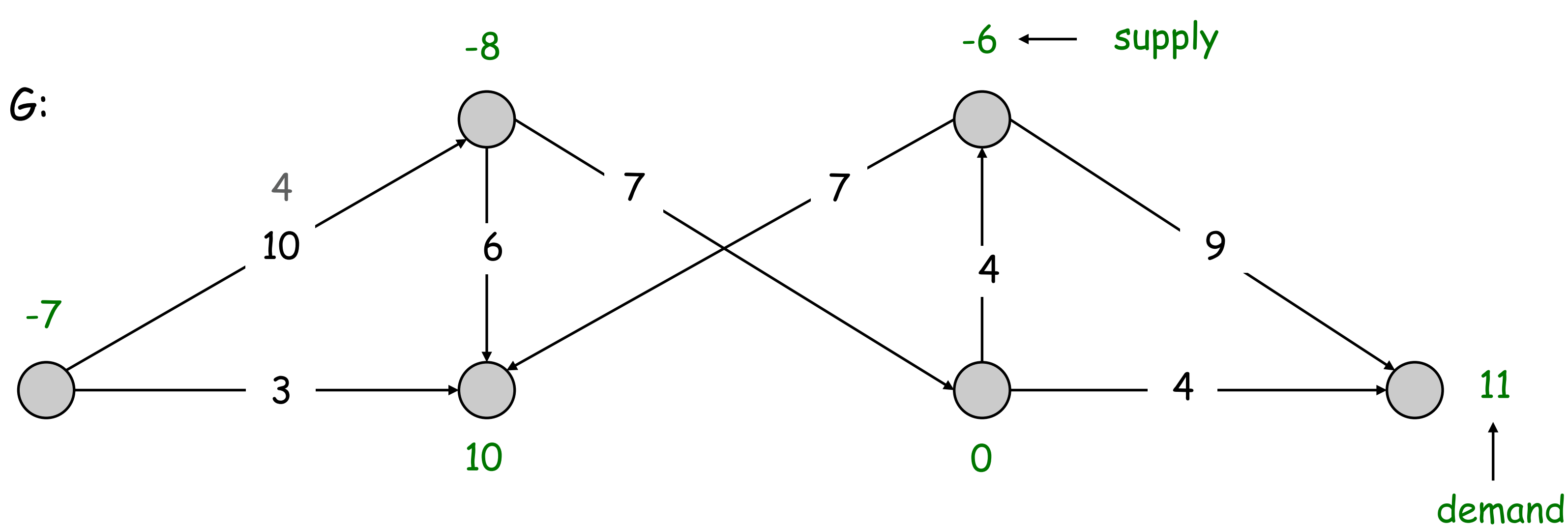
$$\sum_{v\,:\,d(v)>0} d(v) \;=\; \sum_{v\,:\,d(v)<0} -d(v) \;=:\; D$$

Pf. Sum conservation constraints for every demand node v.



Circulation with Demands

Max flow formulation.

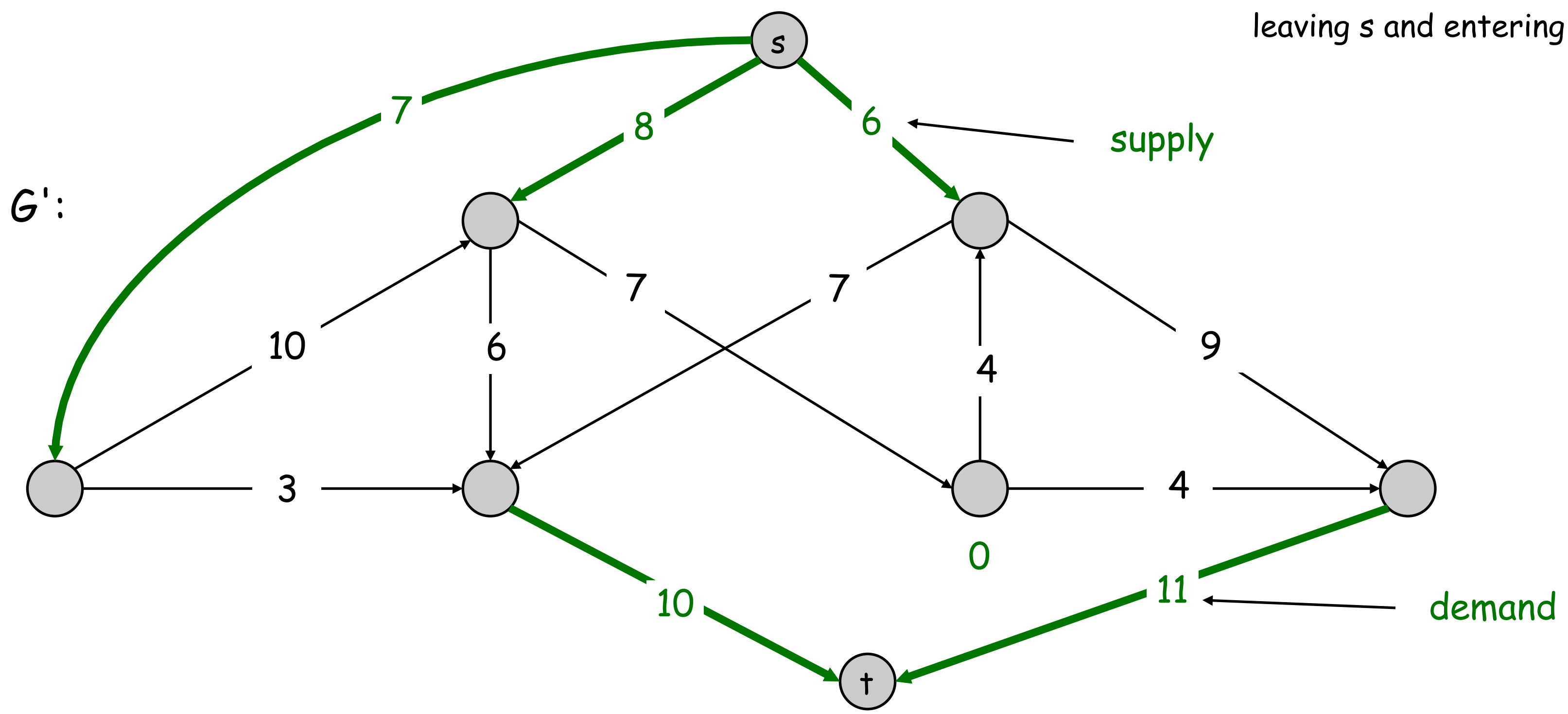


Circulation with Demands

Max flow formulation.

- Add new source s and sink t.
- For each v with d(v) < 0, add edge (s, v) with capacity -d(v).
- For each v with d(v) > 0, add edge (v, t) with capacity d(v).
- Claim: G has circulation iff G' has max flow of value D.

saturates all edges leaving s and entering t



Circulation with Demands

Integrality theorem. If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.

Pf. Follows from max flow formulation and integrality theorem for max flow.

Characterization. Given (V, E, c, d), there does not exists a circulation iff there exists a node partition (A, B) such that $\Sigma_{v \in B} d_v > cap(A, B)$

demand by nodes in B exceeds supply of nodes in B plus max capacity of edges going from A to B

Pf idea. Look at min cut in G'.

Circulation with Demands and Lower Bounds

Feasible circulation.

- Directed graph $G = (V, E)$.
- Edge capacities $c(e)$ and lower bounds $\ell(e)$, $e \in E$.
- Node supply and demands $d(v)$, $v \in V$.

Def. A circulation is a function that satisfies:

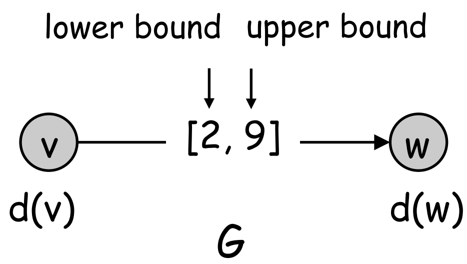
- For each $e \in E$: $\quad \ell(e) \le f(e) \le c(e)$ (capacity)
- For each $v \in V$: $\quad \sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

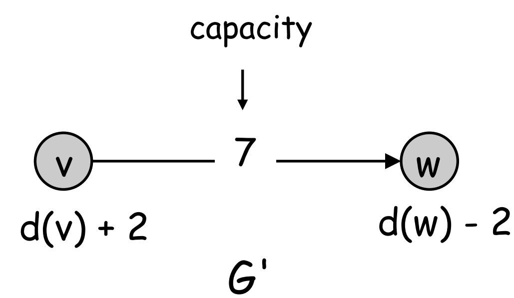
Circulation problem with lower bounds. Given (V, E, ℓ, c, d), does there exists a a circulation?

Circulation with Demands and Lower Bounds

Idea. Model lower bounds with demands.

- Send $\ell(e)$ units of flow along edge e.
- Update demands of both endpoints.





Theorem. There exists a circulation in G iff there exists a circulation in G'. If all demands, capacities, and lower bounds in G are integers, then there is a circulation in G that is integer-valued.

Pf sketch. f(e) is a circulation in G iff f'(e) = f(e) - $\ell(e)$ is a circulation in G'.

7.8 Survey Design

Survey Design

Survey design.

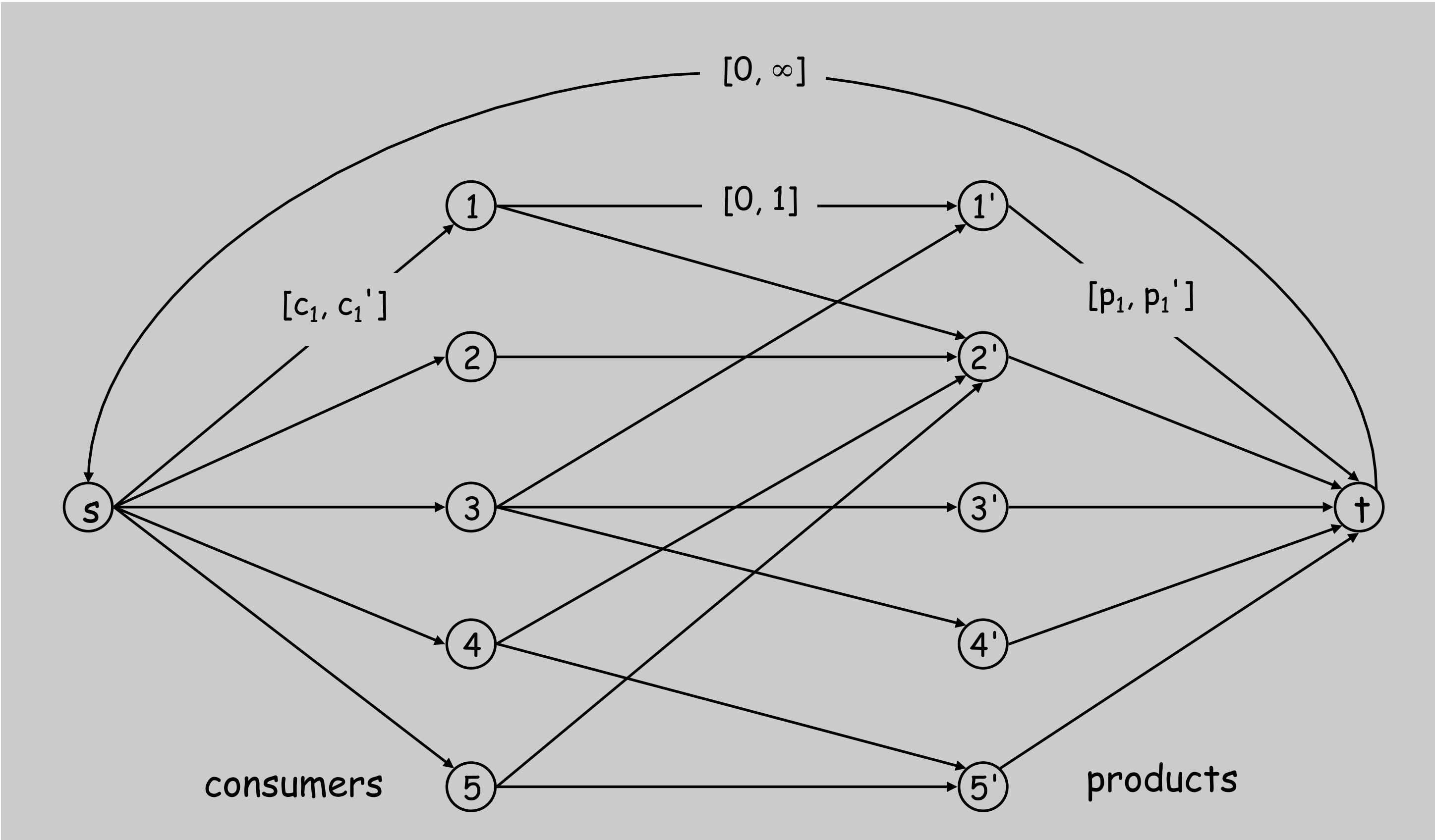
- Design survey asking n_1 consumers about n_2 products.
- Can only survey consumer i about a product j if they own it.
- Ask consumer i between c_i and c_i' questions.
- Ask between p_j and p_j' consumers about product j.

Goal. Design a survey that meets these specs, if possible.

Bipartite perfect matching. Special case when $c_i = c_i' = p_i = p_i' = 1$.

Survey Design

Algorithm. Formulate as a circulation problem with lower bounds.
- Include an edge (i, j) if customer own product i.
- Integer circulation $\Leftrightarrow$ feasible survey design.



7.10 Image Segmentation

Image Segmentation

Image segmentation.

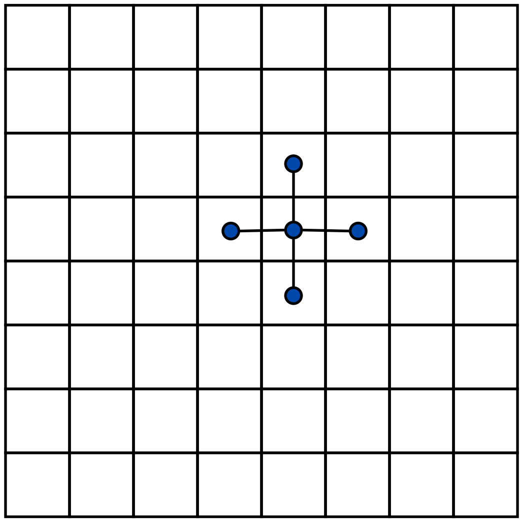
- Central problem in image processing.
- Divide image into coherent regions.

Ex: Three people standing in front of complex background scene. Identify each person as a coherent object.

Image Segmentation

Foreground / background segmentation.

- Label each pixel in picture as belonging to foreground or background.
- V = set of pixels, E = pairs of neighboring pixels.
- $a_i \geq 0$ is likelihood pixel i in foreground.
- $b_i \geq 0$ is likelihood pixel i in background.
- $p_{ij} \geq 0$ is separation penalty for labeling one of i and j as foreground, and the other as background.



Goals.

- Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground.
- Smoothness: if many neighbors of i are labeled foreground, we should be inclined to label i as foreground.
- Find partition (A, B) that maximizes: $\displaystyle \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$

  foreground (A) background (B)

Image Segmentation

Formulate as min cut problem.

- Maximization.
- No source or sink.
- Undirected graph.

Turn into minimization problem.

- Maximizing $$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

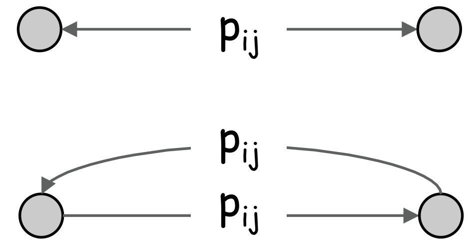
  is equivalent to minimizing $$\underbrace{\left(\sum_{i \in V} a_i + \sum_{j \in V} b_j\right)}_{\text{a constant}} - \sum_{i \in A} a_i - \sum_{j \in B} b_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

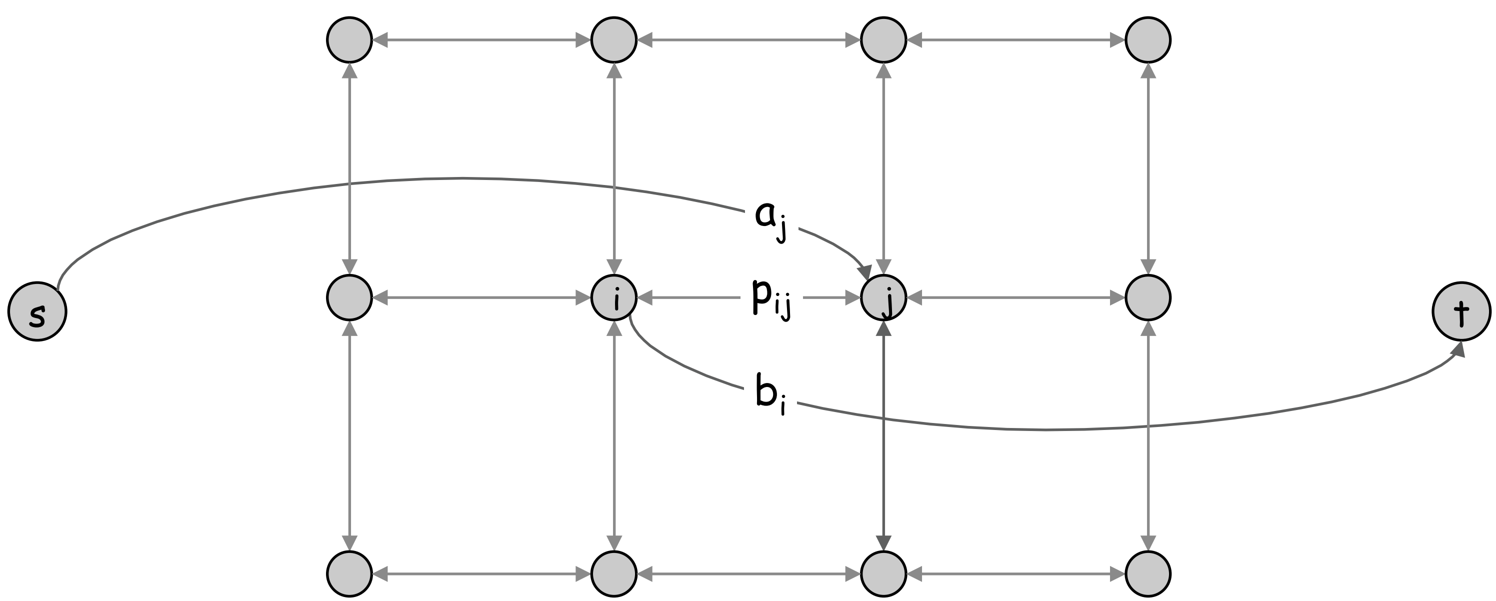
- or alternatively $$\sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

Image Segmentation

Formulate as min cut problem.

- $G' = (V', E')$.
- Add source to correspond to foreground; add sink to correspond to background
- Use two anti-parallel edges instead of undirected edge.





G'

Image Segmentation

Consider min cut (A, B) in G'.

- A = foreground.

$$cap(A,B) \;=\; \sum_{j \in B} a_j + \sum_{i \in A} b_i \;+ \sum_{\substack{(i,j) \in E \\ i \in A,\; j \in B}} p_{ij}$$

if i and j on different sides, p_{ij} counted exactly once

- Precisely the quantity we want to minimize.