CSCI 740 - Programming Language Theory

Lecture 6

The λ-calculus

Instructor: Hossein Hojjat

September 11, 2017

Programming Language Features

Many features in programming languages are not essential: they are for convenience

• Multi-argument functions • Use Currying	`def f(x, y, z)`
• Loops • Use recursion	`while (x == y) {...}`
• Side effects • Use functional programming	`var (x = 1)`

- What languages features are really necessary?

Core Language

- Goal: Come up with a "core" language
  - As small as possible
  - Still Turing complete
- Gives an opportunity of studying important language features

Core Language

- Goal: Come up with a "core" language
  - As small as possible
  - Still Turing complete
- Gives an opportunity of studying important language features

λ-calculus

- Peter J Landin (1960s):
  λ-calculus can be used to model a complex programming language
- A programming language is nothing more than λ-calculus plus some syntactic sugar

- "A Correspondence Between ALGOL 60 and Church's Lambda-Notation"(1965)
- "The Next 700 Programming Languages"(1966)

- λ-calculus plays a similar role for PL research as Turing machines do for computability

Lambda Expressions

- A λ-calculus expression is defined as

$$\begin{aligned} e \;::=\; & x && \text{variable} \\ \mid\; & \lambda x.e && \text{abstraction} \\ \mid\; & e\ e && \text{application} \end{aligned}$$

Application

$$e_1 \;\; e_2$$

function — argument

- Application is left associative $(e_1\ e_2\ e_3\ e_4) \equiv (((e_1\ e_2)\ e_3)\ e_4)$

Abstraction

$$\lambda\ x\ .\ e$$

binder — body

- Scope of the dot in an abstraction extends as far to the right as possible

$\lambda x.e_1\ e_2\ e_3$ means $(\lambda x.(e_1\ e_2\ e_3))$ and not $((\lambda x.e_1\ e_2)\ e_3)$

Examples of Lambda Expressions

- The identity function

Examples of Lambda Expressions

- The identity function

$$\lambda x.x$$

Examples of Lambda Expressions

- The identity function

$$\lambda x.x$$

- Function that takes two other functions and produces their composition

Examples of Lambda Expressions

- The identity function

$$\lambda x.x$$

- Function that takes two other functions and produces their composition

$$\lambda f.\lambda g.(f\ g)$$

Examples of Lambda Expressions

- The identity function

$$\lambda x.x$$

- Function that takes two other functions and produces their composition

$$\lambda f.\lambda g.(f\ \ g)$$

- Function that flips the order of two arguments that are passed to a function

Examples of Lambda Expressions

- The identity function

$$\lambda x.x$$

- Function that takes two other functions and produces their composition

$$\lambda f.\lambda g.(f\ g)$$

- Function that flips the order of two arguments that are passed to a function

$$\lambda f.\lambda x.\lambda y.(f\ y\ x)$$

Multiple Arguments

- λ-calculus provides no built-in support for multi-argument functions
- Do we need an extension $\lambda(x_1 \cdots x_n)\ e$ to support multiple arguments?

Multiple Arguments

- λ-calculus provides no built-in support for multi-argument functions
- Do we need an extension $\lambda(x_1 \cdots x_n)\ e$ to support multiple arguments?
- **Currying:** reducing a function with multiple arguments to functions with a single argument

$$(\lambda(x_1 \cdots x_n)\ e) \ \Rightarrow\ (\lambda x_1(\lambda \cdots (\lambda x_n\ e) \cdots)$$

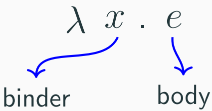
Example.

- Consider the function $\lambda(x, y).(x\ y)$
- Curried form: $f = \lambda x.\lambda y.(x\ y)$
- $g = (f\ \lambda x.x)$ is a function that takes a single argument and applies identity to it

Static Scoping

- Scope of a variable:
  portion of program where the identifier is accessible
- An abstraction $\lambda x.e$ binds the variable x in e
  - e is the scope of x
  - x is bound in $\lambda x.e$
- λ-calculus uses static scoping
  - Scope of variable is fixed at compile-time to the smallest block containing the variable declaration
- $(\lambda x.x\ (\lambda x.x))\ z$
  - Rightmost x refers to the second binding
  - Takes its argument and applies it to the identity function

Free and Bound Variables



- Occurrences of x in the body e are bound
- Nonbound variable occurrences are called free

Free Variables

$$
\begin{aligned}
\mathsf{FV}(x) &= \{x\} \\
\mathsf{FV}(e_1\ e_2) &= \mathsf{FV}(e_1) \cup \mathsf{FV}(e_2) \\
\mathsf{FV}(\lambda x.e) &= \mathsf{FV}(e) - \{x\}
\end{aligned}
$$

Free and Bound Variables

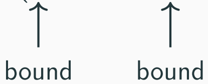
$$\lambda x.\big(x \;\; (\lambda y.y \;\; z) \;\; x\big) \;\; y$$

Free and Bound Variables

$$\lambda x.\Big(\underset{\substack{\uparrow \\ \text{bound}}}{x} \quad (\lambda y.y \quad z) \quad x\Big) \quad y$$

Free and Bound Variables

$$\lambda x.(x \;\; (\lambda y.y \;\; z) \;\; x) \;\; y$$



Free and Bound Variables

$$\lambda x.\Big(\underset{\text{bound}}{\underset{\uparrow}{x}} \;\; (\lambda y.\underset{\text{bound}}{\underset{\uparrow}{y}} \;\; \underset{\text{free}}{\underset{\uparrow}{z}}) \;\; x\Big) \;\; y$$

Free and Bound Variables

$$\lambda x.\big(\underset{\text{bound}}{\underset{\uparrow}{x}}\ \ (\lambda y.\underset{\text{bound}}{\underset{\uparrow}{y}}\ \ \underset{\text{free}}{\underset{\uparrow}{z}})\ \ \underset{\text{bound}}{\underset{\uparrow}{x}}\big)\ \ y$$

Free and Bound Variables

$$\lambda x.\Big(\underset{\substack{\uparrow\\ \text{bound}}}{x}\ \ (\lambda y.\underset{\substack{\uparrow\\ \text{bound}}}{y}\ \ \underset{\substack{\uparrow\\ \text{free}}}{z})\ \ \underset{\substack{\uparrow\\ \text{bound}}}{x}\Big)\ \ \underset{\substack{\uparrow\\ \text{free}}}{y}$$

- An expression is **closed** if all variables are bound
- An expression is **open** if it is not closed

Free and Bound Variables

- Like in any language with statically nested scoping, we need to worry about variable shadowing
- Shadowing: a variable declared within a scope has the same name as a variable declared in an outer scope

$$\lambda x.\ \ x\ \ (\lambda x.x)\ \ x$$

Renaming Bound Variables

- **α-equivalence**: λ-expressions that can be obtained from one another by renaming the bound variables are identical
- **Example:** $\lambda x.x$ is identical to $\lambda y.y$ and to $\lambda z.z$
- **Intuition:**
  Change the name of a formal argument and of all its occurrences in the function body: function behavior does not change
- α-conversion: renaming the bound variables in the expression
- Always try to rename bound variables so that they are all unique
  - Write $\lambda x.x\ (\lambda y.y)\ x$ instead of $\lambda x.x\ (\lambda x.x)\ x$
  - Makes it easy to see the scope of bindings

Substitution

$e[x \mapsto e_1]$

λ-expression obtained by replacing each free occurrence of the variable x in e by the λ-expression e_1

- **Valid** Substitution: no free variable in e_1 becomes bound as a result of the substitution $e[x \mapsto e_1]$
- **Invalid** Substitution: involves a variable capture or a name clash

Example:

- an invalid substitution:

$$(\lambda x.x\ y)[y \mapsto x] \stackrel{?}{=} \lambda x.x\ x$$

- first function: applies argument to y
- second function: identity

Substitution

- Substitution is defined by cases on $e[x \mapsto e']$

Variable

$$y[x \mapsto e'] = e' \qquad \text{if } x = y$$
$$y[x \mapsto e'] = y \qquad \text{otherwise}$$

Application

$$(e_1 \;\; e_2)[x \mapsto e'] = (e_1[x \mapsto e'] \;\; e_2[x \mapsto e'])$$

Abstraction

$$(\lambda y.e_1)[x \mapsto e'] = \lambda y.e_1 \qquad \text{if } x = y$$
$$(\lambda y.e_1)[x \mapsto e'] = \lambda z.((e_1[y \mapsto z])[x \mapsto e']) \qquad \text{otherwise}$$
$$\text{where } z \notin \left(\mathsf{FV}(e_1) \cup \mathsf{FV}(e') \cup \{x\}\right)$$

β-reduction

- In λ-calculus there is one computation rule called β-reduction

$$(\lambda x.e)\ e_1 \quad \to_\beta \quad e[x \mapsto e_1]$$

- A β-redex (<u>red</u>ucible <u>ex</u>pression) is a term of form $(\lambda x.e)\ e_1$
- λ-definable functions coincide with the Turing-computable functions (λ-calculus is Turing-complete)
- Any possible computation can be defined in terms of β-reduction rule

β-reduction Exercise

- Apply the β-reductions in the following expression

$$((\lambda x.\lambda y.x)\ y)\ z$$

β-reduction Exercise

- Apply the β-reductions in the following expression

$((\lambda x.\lambda y.x)\ y)\ z$

$\rightarrow_\beta$	$((\lambda y.x)[x \mapsto y])\ z$	substitute y for x in the body of $\lambda y.x$
$\rightarrow_\alpha$	$((\lambda y'.x)[x \mapsto y])\ z$	after α-conversion
$\rightarrow$	$(\lambda y'.y)\ z$	first β-reduction complete
$\rightarrow_\beta$	$y[y' \mapsto z]$	substitute z for y' in y
$\rightarrow$	y	second β-reduction complete

β-reduction Exercise

- Apply the β-reductions in the following expression

$$(\lambda x.x\ x)(\lambda x.x\ x)$$

β-reduction Exercise

- Apply the β-reductions in the following expression

$$(\lambda x.x\ x)(\lambda x.x\ x)$$

- Each time you beta-reduce you get the same expression back
- λ-calculus is equivalent to Turing machine
- Turing machine may fail to halt

η-conversion

- η-reduction is useful to eliminate redundant λ-abstractions

$$\lambda x.(e\ x) \ \rightarrow_\eta \ e \qquad \text{if } x \notin \mathsf{FV}(e)$$

- Motivation for η-conversion:
- $\lambda x.(e\ x)$ and e behave identically as functions

$$(\lambda x.(e\ x))\ u \quad \rightarrow_\beta \quad (e\ u) \qquad \text{if } x \notin \mathsf{FV}(e)$$

Example:

$$\begin{aligned} & (\lambda x.(\lambda y.x\ y)\ (y\ w)) \\ \rightarrow_\alpha \quad & (\lambda x.(\lambda y'.x\ y')\ (y\ w)) \\ \rightarrow_\beta \quad & \lambda y'.((y\ w)\ y') \\ \rightarrow_\eta \quad & (y\ w) \end{aligned}$$

Computing with λ-calculus

- λ-calculus is a **core** programming language: it is Turing complete
- How can we possibly compute with the λ-calculus when we have no data to manipulate?

1. no numbers
2. no data-structures
3. no control structures (if-then-else, loops)

Computing with λ-calculus

- λ-calculus is a **core** programming language: it is Turing complete
- How can we possibly compute with the λ-calculus when we have no data to manipulate?

1. no numbers
2. no data-structures
3. no control structures (if-then-else, loops)

Church Encoding:

Representation of data and operators in the lambda calculus

- Not intended as a practical implementation of primitive data types
- It shows that other primitive data types are not required to represent any calculation

Encoding Booleans

- We need to define functions TRUE, FALSE, AND, NOT, IF that behave as expected
- For example:

$$\begin{aligned}\text{AND TRUE FALSE} &= \text{FALSE}\\ \text{NOT FALSE} &= \text{TRUE}\\ \text{IF TRUE } e_1\ e_2 &= e_1\\ \text{IF FALSE } e_1\ e_2 &= e_2\end{aligned}$$

Encoding Booleans

- We can define TRUE and FALSE as e.g. following

$$\text{TRUE} \triangleq \lambda x.\lambda y.x$$
$$\text{FALSE} \triangleq \lambda x.\lambda y.y$$

- Both TRUE and FALSE take two arguments:

- TRUE returns the first
- FALSE returns the second

Encoding Booleans

- Conditional statement takes three arguments b,e_t,e_f where: b is a Boolean value and e_t , e_f are arbitrary λ-expressions

$$\text{if} = \lambda b e_t e_f . \begin{cases} e_t & \text{if } b = \text{true} \\ e_f & \text{if } b = \text{false} \end{cases}$$

Encoding Booleans

- Conditional statement takes three arguments b,e_t,e_f where: b is a Boolean value and e_t , e_f are arbitrary λ-expressions

$$\text{if} = \lambda b e_t e_f . \begin{cases} e_t & \text{if } b = \text{true} \\ e_f & \text{if } b = \text{false} \end{cases}$$

- Since TRUE e_t e_f $\rightarrow_\beta$ e_t and FALSE e_t e_f $\rightarrow_\beta$ e_f we define

$$\text{if} \triangleq \lambda b e_t e_f . b \; e_t \; e_f$$

Encoding Booleans

$$\text{NOT} \triangleq \lambda b.b \ \text{FALSE TRUE}$$
$$\text{AND} \triangleq \lambda b_1.\lambda b_2.b_1 \ b_2 \ \text{FALSE}$$
$$\text{OR} \triangleq \lambda b_1.\lambda b_2.b_1 \ \text{TRUE} \ b_2$$

Church Numerals

“0”	$\lambda s.\lambda z.z$
“1”	$\lambda s.\lambda z.s\ z$
“2”	$\lambda s.\lambda z.s\ (s\ z)$
“3”	$\lambda s.\lambda z.s\ (s\ (s\ z))$
$\ldots$	

- Encode numbers with two-argument functions
- “Number i” composes the first argument i times, starting with the second argument
- z stands for “zero” and s for “successor” (think unary)

Church Numerals

"0"	$\lambda s.\lambda z.z$
"1"	$\lambda s.\lambda z.s\ z$
"2"	$\lambda s.\lambda z.s\ (s\ z)$
"3"	$\lambda s.\lambda z.s\ (s\ (s\ z))$
$\ldots$	
"successor"	$\lambda n.\lambda s.\lambda z.s\ (n\ s\ z)$

- successor: take "a number" and return "a number" that (when called) applies s one more time