CSCI 740 - Programming Language Theory

Lecture 6

The λ-calculus

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Programming Language Features

Many features in programming languages are not essential: they are for convenience

• Multi-argument functions • Use Currying	`def f(x, y, z)`
• Loops • Use recursion	`while (x == y) {...}`
• Side effects • Use functional programming	`var (x = 1)`

- What languages features are really necessary?

Core Language

- Goal: Come up with a "core" language
  - As small as possible
  - Still Turing complete
- Gives an opportunity of studying important language features

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λ-calculus

- Peter J Landin (1960s):
  λ-calculus can be used to model a complex programming language
- A programming language is nothing more than λ-calculus plus some syntactic sugar

- "A Correspondence Between ALGOL 60 and Church's Lambda-Notation"(1965)
- "The Next 700 Programming Languages"(1966)

- λ-calculus plays a similar role for PL research as Turing machines do for computability

Lambda Expressions

- A λ-calculus expression is defined as

$$\begin{array}{rcll} e & ::= & x & \text{variable} \\ & \mid & \lambda x.e & \text{abstraction} \\ & \mid & e\ e & \text{application} \end{array}$$

Application

$$\underbrace{e_1}_{\text{function}} \quad \underbrace{e_2}_{\text{argument}}$$

- Application is left associative $(e_1\ e_2\ e_3\ e_4) \equiv (((e_1\ e_2)\ e_3)\ e_4)$

Abstraction

$$\lambda\ \underbrace{x}_{\text{binder}}\ .\ \underbrace{e}_{\text{body}}$$

- Scope of the dot in an abstraction extends as far to the right as possible

$$\lambda x.e_1\ e_2\ e_3 \quad \text{means} \quad (\lambda x.(e_1\ e_2\ e_3)) \quad \text{and not} \quad ((\lambda x.e_1\ e_2)\ e_3)$$

Examples of Lambda Expressions

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- Function that flips the order of two arguments that are passed to a function

$$\lambda f.\lambda x.\lambda y.(f\ y\ x)$$

Multiple Arguments

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- Do we need an extension $\lambda(x_1 \cdots x_n)\ e$ to support multiple arguments?

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- λ-calculus provides no built-in support for multi-argument functions
- Do we need an extension $\lambda(x_1 \cdots x_n)\ e$ to support multiple arguments?
- **Currying:** reducing a function with multiple arguments to functions with a single argument

$$(\lambda(x_1 \cdots x_n).e) \;\Rightarrow\; (\lambda x_1.(\lambda \cdots (\lambda x_n.e) \cdots)$$

Example.

- Consider the function $\lambda(x, y).(x\ y)$
- Curried form: $f = \lambda x.\lambda y.(x\ y)$
- $g = (f\ \lambda x.x)$ is a function that takes a single argument and applies identity to it

Static Scoping

- Scope of a variable:
  portion of program where the identifier is accessible
- An abstraction $\lambda x.e$ binds the variable x in e
  - e is the scope of x
  - x is bound in $\lambda x.e$
- λ-calculus uses static scoping
  - Scope of variable is fixed at compile-time to the smallest block containing the variable declaration
- $(\lambda x.x\ (\lambda x.x))\ z$
  - Rightmost x refers to the second binding
  - Takes its argument and applies it to the identity function

Free and Bound Variables

$$\lambda\ \underbrace{x}_{\text{binder}}\ .\ \underbrace{e}_{\text{body}}$$

- Occurrences of x in the body e are bound
- Nonbound variable occurrences are called free

Free Variables

$$
\begin{aligned}
\mathsf{FV}(x) &= \{x\} \\
\mathsf{FV}(e_1\ e_2) &= \mathsf{FV}(e_1) \cup \mathsf{FV}(e_2) \\
\mathsf{FV}(\lambda x.e) &= \mathsf{FV}(e) - \{x\}
\end{aligned}
$$

Free and Bound Variables

$$\lambda x.\big(x \;\; (\lambda y.y \;\; z) \;\; x\big) \;\; y$$

Free and Bound Variables

$$\lambda x.\Big(\underset{\substack{\uparrow \\ \text{bound}}}{x}\ \ (\lambda y.y\ \ z)\ \ x\Big)\ \ y$$

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$$\lambda x.(\underset{\substack{\uparrow \\ \text{bound}}}{x} \;\; (\lambda y.\underset{\substack{\uparrow \\ \text{bound}}}{y} \;\; \underset{\substack{\uparrow \\ \text{free}}}{z}) \;\; x) \;\; y$$

Free and Bound Variables

$$\lambda x.\Big(\underset{\substack{\big\uparrow\\\text{bound}}}{x}\ \ (\lambda y.\underset{\substack{\big\uparrow\\\text{bound}}}{y}\ \ \underset{\substack{\big\uparrow\\\text{free}}}{z})\ \ \underset{\substack{\big\uparrow\\\text{bound}}}{x}\Big)\ \ y$$

Free and Bound Variables

$$\lambda x.(\underset{\text{bound}}{\underset{\uparrow}{x}} \quad (\lambda y.\underset{\text{bound}}{\underset{\uparrow}{y}} \quad \underset{\text{free}}{\underset{\uparrow}{z}}) \quad \underset{\text{bound}}{\underset{\uparrow}{x}}) \quad \underset{\text{free}}{\underset{\uparrow}{y}}$$

- A λ-expression without free variables is **closed** or a **combinator**
- An expression is **open** if it is not closed

Free and Bound Variables

$$\lambda x.(\underset{\text{bound}}{\underset{\uparrow}{x}} \quad (\lambda y.\underset{\text{bound}}{\underset{\uparrow}{y}} \quad \underset{\text{free}}{\underset{\uparrow}{z}}) \quad \underset{\text{bound}}{\underset{\uparrow}{x}}) \quad \underset{\text{free}}{\underset{\uparrow}{y}}$$

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Example.

Some interesting combinators (we will discuss these combinators more)

$$\begin{aligned} \Omega &= \ (\lambda x.x\ x)(\lambda x.x\ x) \\ Y &= \ \lambda f.(\lambda x.f(x\ x))\ (\lambda x.f(x\ x)) \end{aligned}$$

Free and Bound Variables

- Like in any language with statically nested scoping,
  we need to worry about variable shadowing
- Shadowing: a variable declared within a scope has the same name as
  a variable declared in an outer scope

$$\lambda x.\ \ x\ \ (\lambda x.x)\ \ x$$

Renaming Bound Variables

- **α-equivalence**: λ-expressions that can be obtained from one another by renaming the bound variables are identical
- **Example:** $\lambda x.x$ is identical to $\lambda y.y$ and to $\lambda z.z$
- **Intuition:**
  Change the name of a formal argument and of all its occurrences in the function body: function behavior does not change
- α-conversion: renaming the bound variables in the expression
- Always try to rename bound variables so that they are all unique
  - Write $\lambda x.x\ (\lambda y.y)\ x$ instead of $\lambda x.x\ (\lambda x.x)\ x$
  - Makes it easy to see the scope of bindings

Substitution

$$\boxed{e[x \mapsto e_1]}$$

λ-expression obtained by replacing each free occurrence of the variable x in e by the λ-expression e_1

- **Valid** Substitution: no free variable in e_1 becomes bound as a result of the substitution $e[x \mapsto e_1]$
- **Invalid** Substitution: involves a variable capture or a name clash

Example:

- an invalid substitution:

$$(\lambda x.x\ y)[y \mapsto x] \stackrel{?}{=} \lambda x.x\ x$$

- first function: applies argument to y
- second function: identity

Substitution

- Substitution is defined by cases on $e[x \mapsto e']$

Variable

$$\begin{aligned} y[x \mapsto e'] &= e' && \text{if } x = y \\ y[x \mapsto e'] &= y && \text{otherwise} \end{aligned}$$

Application

$$(e_1 \quad e_2)[x \mapsto e'] = (e_1[x \mapsto e'] \quad e_2[x \mapsto e'])$$

Abstraction

$$\begin{aligned} (\lambda y.e_1)[x \mapsto e'] &= \lambda y.e_1 && \text{if } x = y \\ (\lambda y.e_1)[x \mapsto e'] &= \lambda z.((e_1[y \mapsto z])[x \mapsto e']) && \text{otherwise} \\ &\quad \text{where } z \notin \big(\mathsf{FV}(e_1) \cup \mathsf{FV}(e') \cup \{x\}\big) \end{aligned}$$

β-reduction

- In λ-calculus there is one computation rule called β-reduction

$$(\lambda x.e)\ e_1 \quad \rightarrow_\beta \quad e[x \mapsto e_1]$$

- A β-redex (<u>red</u>ucible <u>ex</u>pression) is a term of form $(\lambda x.e)\ e_1$
- λ-definable functions coincide with the Turing-computable functions (λ-calculus is Turing-complete)
- Any possible computation can be defined in terms of β-reduction rule

β-reduction Exercise

- Apply the β-reductions in the following expression

$((\lambda x.\lambda y.x)\ y)\ z$

β-reduction Exercise

- Apply the β-reductions in the following expression

$((\lambda x.\lambda y.x)\ y)\ z$

$\rightarrow_\beta$	$((\lambda y.x)[x \mapsto y])\ z$	substitute y for x in the body of $\lambda y.x$
$\rightarrow_\alpha$	$((\lambda y'.x)[x \mapsto y])\ z$	after α-conversion
$\rightarrow$	$(\lambda y'.y)\ z$	first β-reduction complete
$\rightarrow_\beta$	$y[y' \mapsto z]$	substitute z for y' in y
$\rightarrow$	y	second β-reduction complete

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$$\Omega = (\lambda x.x\ x)(\lambda x.x\ x)$$

β-reduction Exercise

- Apply the β-reductions in the following expression

$$\Omega = (\lambda x.x\ x)(\lambda x.x\ x)$$

- Each time you beta-reduce you get the same expression back
- λ-calculus is equivalent to Turing machine
- Turing machine may fail to halt

η-conversion

- η-reduction is useful to eliminate redundant λ-abstractions

$$\lambda x.(e\ x) \ \to_\eta \ e \qquad\qquad \text{if } x \notin \mathsf{FV}(e)$$

- Motivation for η-conversion:
- $\lambda x.(e\ x)$ and e behave identically as functions

$$(\lambda x.(e\ x))\ u \quad \to_\beta \quad (e\ u) \qquad\qquad \text{if } x \notin \mathsf{FV}(e)$$

Example:

$$\begin{aligned}
& (\lambda x.(\lambda y.x\ y)\ (y\ w)) \\
\to_\alpha \quad & (\lambda x.(\lambda y'.x\ y')\ (y\ w)) \\
\to_\beta \quad & \lambda y'.((y\ w)\ y') \\
\to_\eta \quad & (y\ w)
\end{aligned}$$

Computing with λ-calculus

- λ-calculus is a **core** programming language: it is Turing complete
- How can we possibly compute with the λ-calculus when we have no data to manipulate?

1. no numbers
2. no data-structures
3. no control structures (if-then-else, loops)

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Church Encoding:

Representation of data and operators in the lambda calculus

- Not intended as a practical implementation of primitive data types
- It shows that other primitive data types are not required to represent any calculation

Encoding Booleans

- We need to define functions TRUE, FALSE, AND, NOT, IF that behave as expected
- For example:

$$
\begin{aligned}
\text{AND TRUE FALSE} &= \text{FALSE} \\
\text{NOT FALSE} &= \text{TRUE} \\
\text{IF TRUE } e_1\ e_2 &= e_1 \\
\text{IF FALSE } e_1\ e_2 &= e_2
\end{aligned}
$$

Encoding Booleans

- We can define TRUE and FALSE as e.g. following

$$\text{TRUE} \triangleq \lambda x.\lambda y.x$$
$$\text{FALSE} \triangleq \lambda x.\lambda y.y$$

- Both TRUE and FALSE take two arguments:

- TRUE returns the first
- FALSE returns the second

Encoding Booleans

- Conditional statement takes three arguments b,e_t,e_f where: b is a Boolean value and e_t , e_f are arbitrary λ-expressions

$$\mathsf{IF} = \lambda b.\lambda e_t.\lambda e_f.\begin{cases} e_t & \text{if } b = \mathsf{true} \\ e_f & \text{if } b = \mathsf{false} \end{cases}$$

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- Since TRUE $e_t\ e_f \ \rightarrow_\beta\ e_t$ and FALSE $e_t\ e_f \ \rightarrow_\beta\ e_f$ we define

$$\text{IF} \triangleq \lambda b.\lambda e_t.\lambda e_f.b\ e_t\ e_f$$

Encoding Booleans

$$\begin{aligned}
\text{NOT} &\triangleq \lambda b.b \text{ FALSE TRUE} \\
\text{AND} &\triangleq \lambda b_1.\lambda b_2.b_1\ b_2 \text{ FALSE} \\
\text{OR} &\triangleq \lambda b_1.\lambda b_2.b_1 \text{ TRUE } b_2
\end{aligned}$$

Church Numerals

$$\begin{array}{ll} c_0 & \lambda s.\lambda z.z \\ c_1 & \lambda s.\lambda z.s\ z \\ c_2 & \lambda s.\lambda z.s\ (s\ z) \\ c_3 & \lambda s.\lambda z.s\ (s\ (s\ z)) \\ \ldots & \end{array}$$

- Encode numbers with two-argument functions
- "Number i" composes the first argument i times, starting with the second argument
- z stands for "zero" and s for "successor" (think unary)

Church Numerals

$$
\begin{array}{rl}
c_0 & \lambda s.\lambda z.z \\
c_1 & \lambda s.\lambda z.s\ z \\
c_2 & \lambda s.\lambda z.s\ (s\ z) \\
c_3 & \lambda s.\lambda z.s\ (s\ (s\ z)) \\
\ldots & \\
\mathsf{SUCC} & \lambda n.\lambda s.\lambda z.s\ (n\ s\ z)
\end{array}
$$

- successor: take "a number" and return "a number" that (when called) applies s one more time