CSCI 740 - Programming Language Theory

Lecture 32

Temporal Logic

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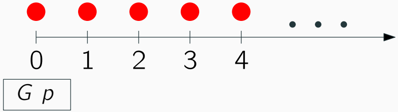
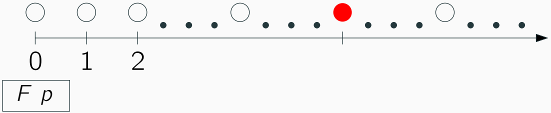
November 17, 2017

Temporal Logic

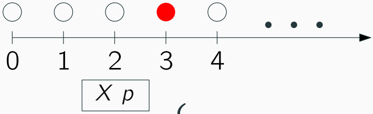
$\mathbf{G\ p}$ is true for a computation path if p holds at all states (points of time)



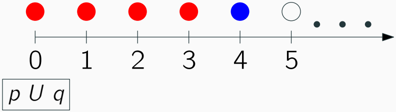
$\mathbf{F\ p}$ is true for a computation path if p holds at some state along that path



$\mathbf{X\ p}$ is true along a path starting in state s_i if p holds in the next state s_{i+1}



$\mathbf{p\ U\ q}$ is true along a path starting at s $\begin{cases} q \text{ is true in some state reachable from } s \\ p \text{ is true in all states from s until q holds} \end{cases}$



Example

What do they mean?

- $\mathbf{G\ F\ p}$

- $\mathbf{F\ G\ p}$

- $\mathbf{G(p \rightarrow F\ q)}$

- $\mathbf{F(p \rightarrow (X\ X\ q))}$

Example

What do they mean?

- $\mathbf{G\ F\ p}$

p holds infinitely often

- $\mathbf{F\ G\ p}$

- $\mathbf{G(p \rightarrow F\ q)}$

- $\mathbf{F(p \rightarrow (X\ X\ q))}$

Example

What do they mean?

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- $\mathbf{F\ G\ p}$

  Eventually, p holds henceforth

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  Every p is eventually followed by a q

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- $\mathbf{F\ G\ p}$

  Eventually, p holds henceforth

- $\mathbf{G(p \to F\ q)}$

  Every p is eventually followed by a q

- $\mathbf{F(p \to (X\ X\ q))}$

  Every p is followed by a q two steps later

Temporal Operators & Relationships

G, F, X, U: All express properties along system traces

- Can you express $G\ p$ purely in terms of F, p, and Boolean operators?

- How about F in terms of U?

- What about X in terms of G, F, or U?

Temporal Operators & Relationships

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Cannot be done

Exercise

Write a temporal logic formula for each of the given properties

- *inv* is true for all states
- In all states it is not the case that *read* and *write*
- At every state a *request* implies that there exists a future point where *grant* holds
- At every state a *request* implies that there exists a future point where *grant* holds, and *request* holds up until that point
- In all states, there is a future position where *enabled* holds
- There is a future position, from which all future positions have *enabled* holding

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$$G\ F\ \mathit{enabled}$$

- There is a future position, from which all future positions have *enabled* holding $F\ G\ \mathit{enabled}$

Liveness vs. Safety

Two terms you are likely to run into:

Safety

- Something bad will never happen

$$G\ \neg \mathit{bad}$$

- If it fails to hold, it's easy to produce a witness

Liveness

- Something good will eventually happen

$$F\ \mathit{good}$$

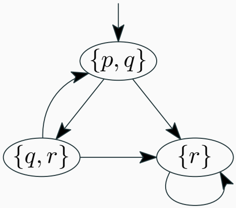
- What does a witness for this look like?

Temporal Logic Flavors

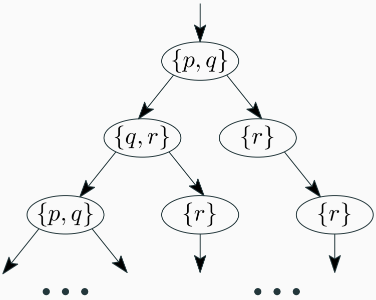
- What we have seen so far are properties expressed over a single computation path or run
- **Linear** Temporal Logic

Computation Tree Logic

- Properties expressed over a tree of all possible executions



Kripke structure



Infinite Computation Tree

Computation Tree Logic

- Computation Tree Logic (CTL, CTL*)
- Properties expressed over a tree of all possible executions
- CTL* gives more expressiveness than LTL
- CTL is a subset of CTL* that is easier to verify than arbitrary CTL*

Computation Tree Logic (CTL*)

- Introduce two extra path quantifiers A and E
- $A\ p$: for all paths f
- $E\ p$: for some path f
- Example:

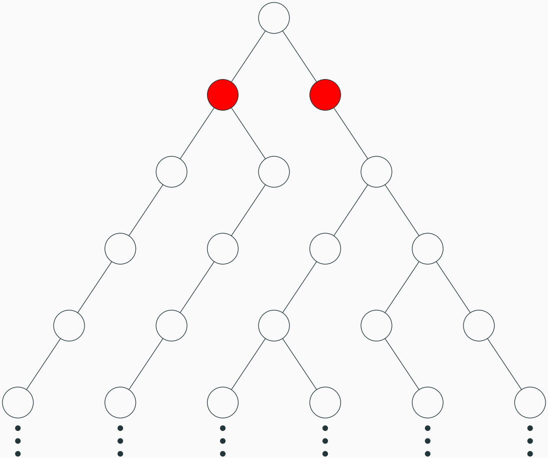
“The grant signal must always be asserted some time after the request signal is asserted”

$$AG(\textit{req} \rightarrow AF\ \textit{grant})$$

Two important subsets:

- LTL : all formulas of the form $A\ p$
  - Example : $A(FG\ p)$
- CTL: there must be a path quantifier before every linear operator
  - Example : $AG(EF\ p)$

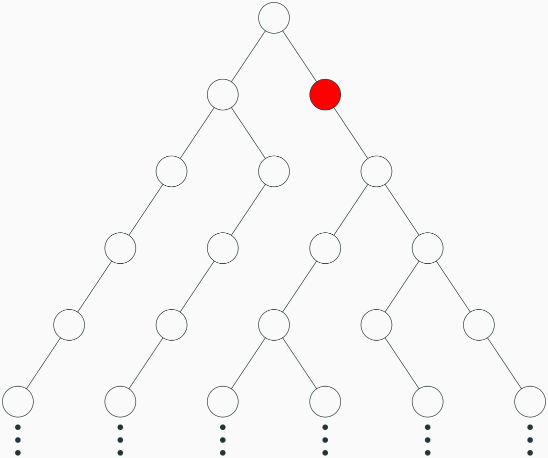
CTL Example



$$AX\ p$$

For every next state p holds

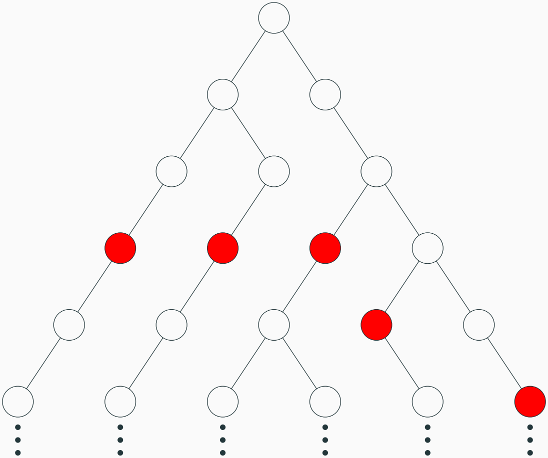
CTL Example



$$EX\ p$$

There exists a next state where p holds

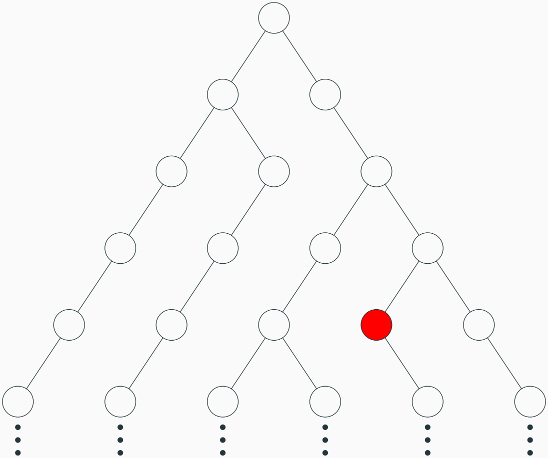
CTL Example



$$AF\ p$$

For all paths, there exists a future state where p holds

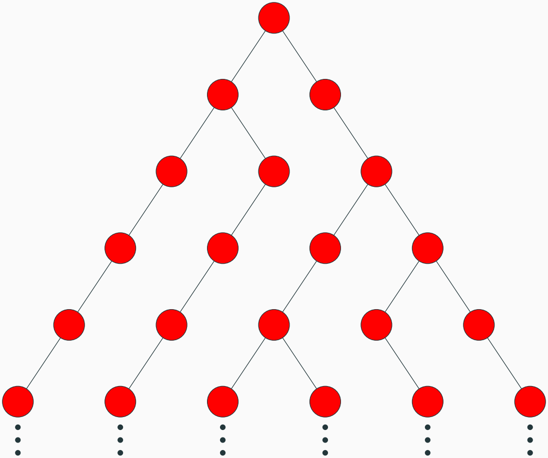
CTL Example



$$EF\ p$$

There exists a path with a future state where p holds

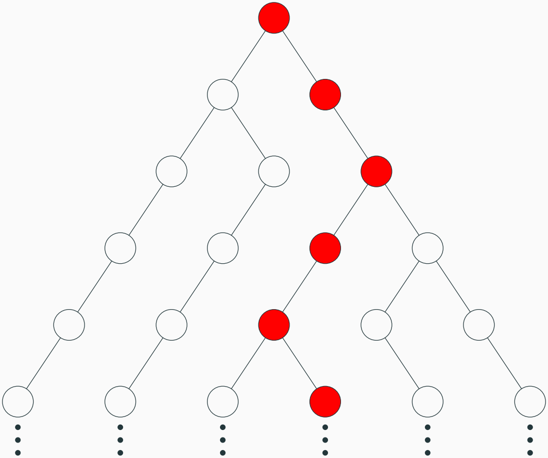
CTL Example



$$AG\ p$$

For all paths, for all states along them, p holds

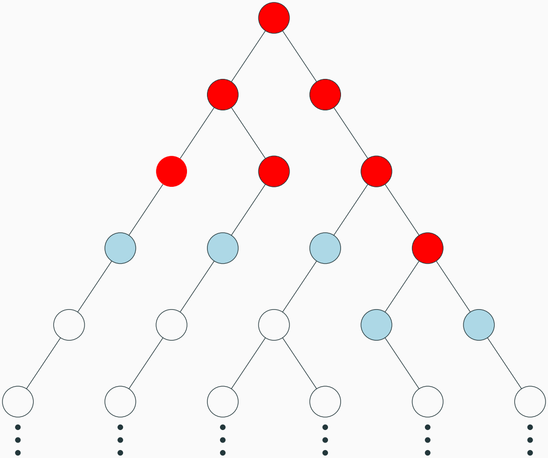
CTL Example



$$EG\ p$$

There exists a path such that, for all states along it, p holds

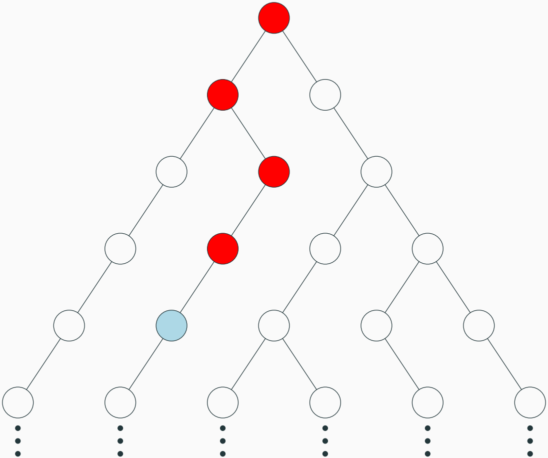
CTL Example



$$A(p\ U\ q)$$

For all paths, q eventually holds, and p holds at all states earlier

CTL Example



$$E(p \; U \; q)$$

Exists path where q eventually holds, and p holds at all states earlier

CTL vs. LTL

- CTL and LTL are not equivalent
- There are properties that can be expressed in LTL but cannot be expressed in CTL
  - For example: $F\ G\ p$
- There are properties that can be expressed in CTL but cannot be expressed in LTL
  - For example: $AG(EF\ p)$
- Hence, expressive power of CTL and LTL are not comparable

Why CTL?

- Verifying LTL properties turns out to be computationally harder than CTL
- But LTL is more intuitive to write
- Complexity of model checking
  - Exponential in the size of the LTL expression
  - Linear for CTL
- For both, model checking is linear in the size of the state graph