CSCI 740 - Programming Language Theory

Lecture 28

Galois Connection

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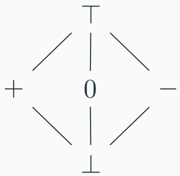
Abstract Domain

- An abstract domain is a lattice
- Elements in the lattice are called abstract values

Example: Sign Abstract Domain



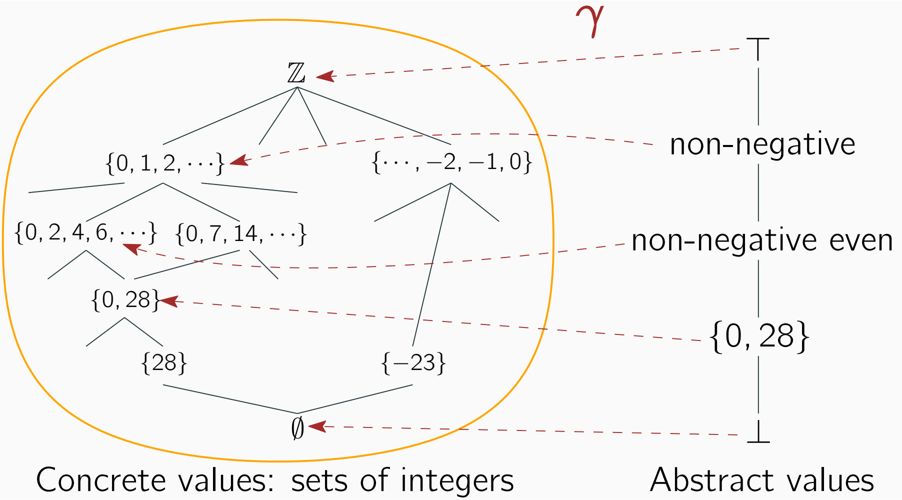
- Set of abstract values $\{\bot, +, 0, -, \top\}$
- Relation $\leq$ that is
  - Reflexive
  - Anti-symmetric
  - Transitive
- Least upper bound (lub, $\sqcup$) and greatest lower bound (glb, $\sqcap$) exists for any pair of elements
  - So it's a lattice

α and γ

Need to relate elements in the lattice with concrete states in the program

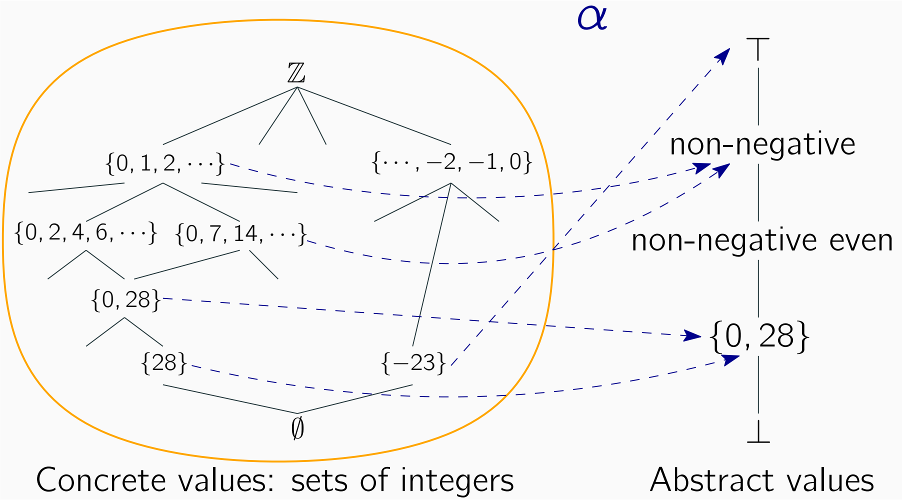
- **Abstraction Function:** $\alpha : 2^C \rightarrow \mathit{Abs}$
  Maps a value in the program to the "best" abstract value
- **Concretization Function:** $\gamma : \mathit{Abs} \rightarrow 2^C$
  Maps an abstract value to a set of values in the program

Abstraction Example



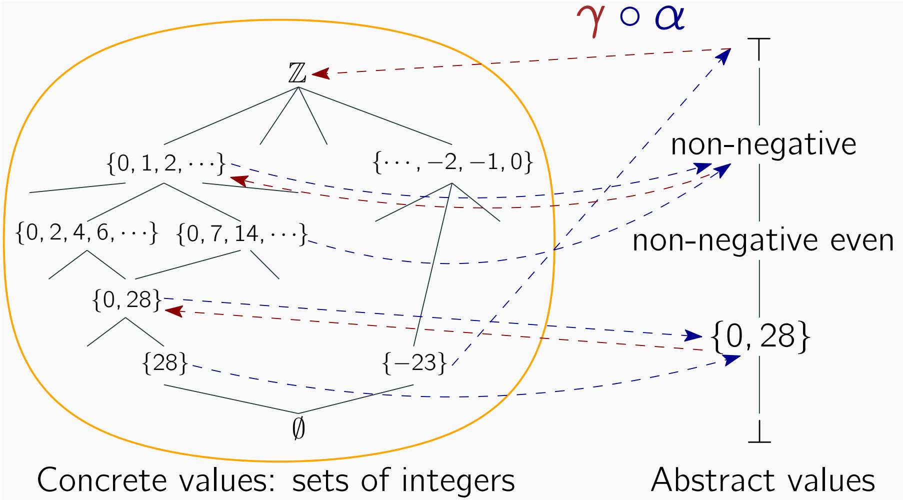
Concretization function γ maps each abstract value to concrete values it represents

Abstraction Example



Abstraction function α maps each concrete set to the best abstract value (least imprecise)

Abstraction Example



Abstraction followed by concretization is sound but imprecise

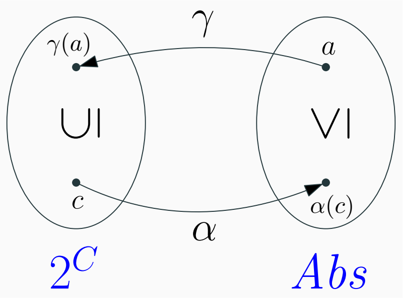
Galois Connection

- α and γ are monotonic (also called "order preserving")
- Given two partial orders $(A_1, \sqsubseteq_1)$ and $(A_2, \sqsubseteq_2)$, we call a function $f : A_1 \to A_2$ monotonic iff for all $x, y \in A_1$:

$$x \sqsubseteq_1 y \to f(x) \sqsubseteq_2 f(y)$$

Galois Connection:

A pair of functions
(Abstraction) $\alpha : 2^C \to \mathit{Abs}$
and
(Concretization) $\gamma : \mathit{Abs} \to 2^C$
such that
$\forall a \in \mathit{Abs}.\ \forall c \in 2^C.\ c \subseteq \gamma(a) \Leftrightarrow \alpha(c) \leq a$



Correctness Conditions

Three conditions guarantee correctness in general:

1. α and γ are monotonic
2. α and γ form a Galois connection
3. Abstract operations op^A are locally correct

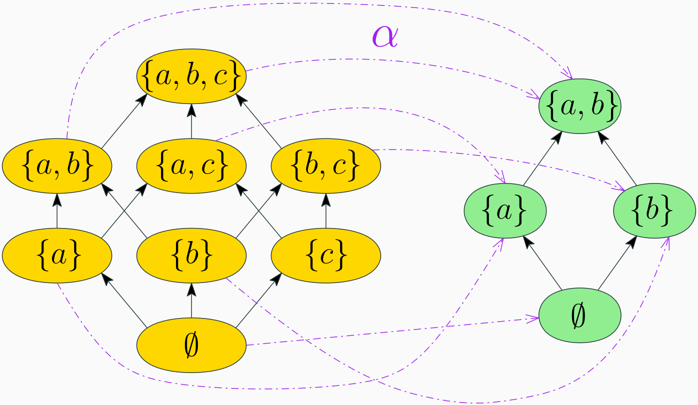
$$\gamma(a_1 \ \mathsf{op}^A \ a_2) \supseteq \gamma(a_1) \ \mathsf{op} \ \gamma(a_2)$$

Key Property of Galois Connections

Theorem. The abstraction and concretization functions uniquely determine each other

$$\gamma(a) = \bigcup \{c \in 2^C \mid \alpha(c) \leq a\}$$
$$\alpha(c) = \bigsqcap \{a \in \mathit{Abs} \mid c \subseteq \gamma(a)\}$$

Example



- $\gamma(\{a\}) = \cup\{\ell \mid \alpha(\ell) \leq \{a\}\} = \emptyset \cup \{a\} \cup \{a,c\} = \{a,c\}$
- $\gamma(\{b\}) = \cup\{\ell \mid \alpha(\ell) \leq \{b\}\} = \emptyset \cup \{b\} \cup \{b,c\} = \{b,c\}$
- $\gamma(\{a,b\}) = \emptyset \cup \{a,b\} \cup \{b,c\} \cup \cdots \cup \{a,b,c\} = \{a,b,c\}$

Galois connection: Intervals and Subsets

Concretization

$$\gamma : [m, n] \to \{x \in \mathbb{Z} \mid m \leq x \leq n\}$$

- m and n are potentially infinite, and $\gamma(\bot) = \emptyset$ and $\gamma(top) = \mathbb{Z}$

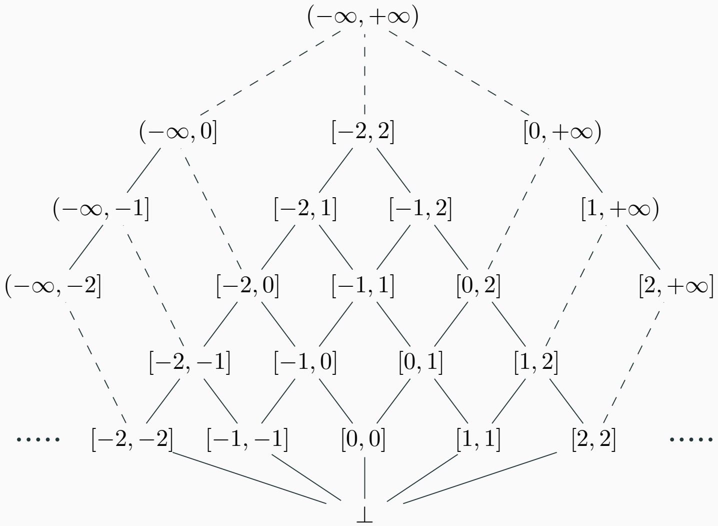
Abstraction

- if S is empty: $\alpha(S) = \bot$, else

$$\alpha : S \to [\textit{inf}\ S, \textit{sup}\ S]$$

- *inf* and *sup* are taken in $\mathbb{Z} \cup \{-\infty, +\infty\}$

Lattice of Intervals



$$[a, b] \sqcup [c, d] = [\mathsf{min}(a, c), \mathsf{max}(b, d)]$$

$$[a, b] \sqcap [c, d] = [\mathsf{max}(a, c), \mathsf{min}(b, d)]$$

Best Abstraction of an Operation

- For a concrete monotone operation $\mathsf{op} : C \to C$ we define an abstract operation $\mathsf{op}^A : A \to A$ by

  $$\mathsf{op}^A(x) = \alpha \circ \mathsf{op} \circ \gamma(x)$$

  It is the best possible abstraction of op

Best Interval Operations

- Let $+$ be the standard interger addition
- What is the best abstraction for intervals?

$$+^A : \text{Inteval} \to \text{Interval}$$

$$\begin{aligned}
[a, b] +^A [c, d] &= \alpha(\gamma([a, b]) + \gamma([c, d])) \\
&= \alpha(\{x \mid a \leq x \leq b\} + \{y \mid c \leq y \leq d\}) \\
&= \alpha(\{x + y \mid a \leq x \leq b, c \leq y \leq d\}) \\
&= \alpha(\{x \mid a + c \leq x \leq b + d\}) \\
&= [a + c, b + d]
\end{aligned}$$

Note.

- We have extended mathematical $+$ to operate over $+\infty$ and $-\infty$
  - e.g. $\forall x \neq -\infty : x + \infty = \infty$

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Exercise. Prove

$$[a,b] \times^A [c,d] = [\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)]$$