CSCI 740 - Programming Language Theory

Lecture 27

Introduction to Abstract Interpretation

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Course Recap

- What we have discussed so far

Operational Semantics

- How will a given program behave on a given input?
- The ground truth for any analysis

Types

- Annotations describe properties of the data that can be referred by a variable
- Easy to describe properties that are global to the execution, but only one variable at a time
  - (at least with the machinery we have seen here)
- Properties are fixed a priori by the type system designer
- Actual analysis is cheap
- Annotations can often be inferred

Program Logics

- Annotations describe properties of the state at a given point in the program
- Easy to describe complex properties of the overall program state, but messy to describe properties that hold over time
- Logic provides a rich language for properties
- Actual analysis can be expensive
- Annotations are hard to infer

Abstract Interpretation

$$W = \mathsf{wp}(\texttt{while } b \texttt{ do } c, Q)$$

- Recall: computing the wp of loop requires solving recursive equation

$$W = (b \Rightarrow \mathsf{wp}(c, W) \land \neg b \Rightarrow Q)$$

- W is the (greatest) fixpoint of the recursive equation
- Abstract Interpretation: interpret a program over an abstract domain
- Find the best possible fixpoint

Patrick Cousot, Radhia Cousot:
"Abstract Interpretation: A Unified Lattice Model for Static Analysis of Programs by Construction or Approximation of Fixpoints", (POPL 1977), pp. 238-252

A Tiny Language

- Consider the following language with only integers and multiplications

$$e \ ::= \ n \mid e_1 \times e_2$$

- Operational semantics of this language:

$$\frac{}{n \Downarrow n} \qquad \frac{e_1 \Downarrow n_1 \qquad e_2 \Downarrow n_2 \qquad n_3 = n_1 \times n_2}{e_1 \times e_2 \Downarrow n_3}$$

- Take the operational semantics as the "ground truth"
- Using operational semantics rules we can define *eval* : $e \rightarrow \mathbb{Z}$
  - *eval*$(n) = n$
  - *eval*$(e_1 \times e_2) =$ *eval*$(e_1) \times$ *eval*(e_2)
- For this language the precise semantics is computable
  - but in general it is not

An Abstraction

- Assume that we are interested not in the value of the expression, but only in its sign
  - positive (+), negative (-), or zero (0)
- We can define an abstract semantics that computes only the sign of the result

$$\mathit{eval}^A : \ e \to \{+, -, 0\}$$

- $\{-, 0, +\}$ is the **abstract domain**

$$\mathit{eval}^A(n) = \mathsf{sign}(n)$$
$$\mathit{eval}^A(e_1 \times e_2) = \mathit{eval}^A(e_1) \otimes \mathit{eval}^A(e_2)$$

$\otimes$	$+$	0	$-$
$+$	$+$	0	$-$
0	0	0	0
$-$	$-$	0	$+$

Soundness

- We can show that this abstraction is correct in the sense that it correctly predicts the sign of an expression
- Proof is by structural induction on e

$$
\begin{aligned}
\mathit{eval}(e) > 0 \;&\Leftrightarrow\; \mathit{eval}^{A}(e) = + \\
\mathit{eval}(e) = 0 \;&\Leftrightarrow\; \mathit{eval}^{A}(e) = 0 \\
\mathit{eval}(e) < 0 \;&\Leftrightarrow\; \mathit{eval}^{A}(e) = -
\end{aligned}
$$

Another View of Soundness

- Associate each abstract value with the set of concrete values it represents
- γ is called the **concretization** function

$$\gamma : \{-, 0, +\} \to 2^{\mathbb{Z}}$$

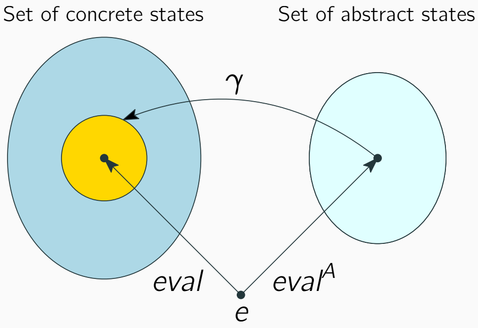
$$\begin{aligned}
\gamma(+) &= \{n \in \mathbb{Z} \mid n > 0\} \\
\gamma(0) &= \{0\} \\
\gamma(-) &= \{n \in \mathbb{Z} \mid n < 0\}
\end{aligned}$$

Another View of Soundness

- Soundness can be stated succinctly

$$\forall e.\ \mathit{eval}(e) \in \gamma(\mathit{eval}^A(e))$$

- The real value of the expression is among the concrete values represented by the abstract value of the expression



Adding $-$

- Extend our language with unary $-$

$$\frac{e \Downarrow n}{-e \Downarrow -n}$$

$$\mathit{eval}^A(-e) = \ominus \mathit{eval}^A(e)$$

	$+$	0	$-$
$\ominus$	$-$	0	$+$

Adding $+$

- Adding addition is not so easy
- The abstract values are not closed under addition

$$\frac{e_1 \Downarrow n_1 \qquad e_2 \Downarrow n_2 \qquad n_3 = n_1 + n_2}{e_1 + e_2 \Downarrow n_3}$$

$$\textit{eval}^A(e_1 + e_2) = \textit{eval}^A(e_1) \oplus \textit{eval}^A(e_2)$$

$\oplus$	$+$	0	$-$
$+$	$+$	$+$	$?$
0	$+$	0	$-$
$-$	$?$	$-$	$-$

Solution

- We need another abstract value to represent a result that can be any integer
- Finding a domain closed under all the abstract operations is often a key design problem

$$\gamma(\top) = \mathbb{Z}$$

$\oplus$	$+$	0	$-$	$\top$
$+$	$+$	$+$	$\top$	$\top$
0	$+$	0	$-$	$\top$
$-$	$\top$	$-$	$-$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

Extending Other Operations

- We also need to extend the other abstract operations to work with $\top$

$\otimes$	$+$	0	$-$	$\top$
$+$	$+$	0	$-$	$\top$
0	0	0	0	0
$-$	$-$	0	$+$	$\top$
$\top$	$\top$	0	$\top$	$\top$

	$+$	0	$-$	$\top$
$\ominus$	$-$	0	$+$	$\top$

Exercise

- We know $(1+2)+-3 \Downarrow 0$
- Compute *eval*$^{A}((1+2)+-3)$

Exercise

- We know $(1+2) + -3 \Downarrow 0$
- Compute $\mathit{eval}^A((1+2) + -3)$

$$
\begin{aligned}
\mathit{eval}^A((1+2) + -3) = \\
& (\mathit{eval}^A(1) \oplus \mathit{eval}^A(2)) \oplus \mathit{eval}^A(-3) = \\
& \qquad (+ \oplus +) \oplus - = \top
\end{aligned}
$$

Loss of Precision

- Abstract computation may lose information

$$\begin{aligned} (1+2)+-3 &\Downarrow 0 \\ \mathit{eval}^A((1+2)+-3) &= \top \end{aligned}$$

- We lost some precision
- But this will simplify the computation of the abstract answer in cases when the precise answer is not computable

Adding Integer Division

- Adding / is straightforward except for the case of division by 0
- If we divide each integer in a set by 0, what set of integers results?
    - The empty set

$$\gamma(\bot) = \{\}$$

$\oslash$	$+$	0	$-$	$\top$	$\bot$
$+$	$+$	0	$-$	$\top$	$\bot$
0	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$-$	$-$	0	$+$	$\top$	$\bot$
$\top$	$\top$	0	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Adding Integer Division

- As before we need to extend the other abstract operations
- In this case, every entry involving bottom is bottom

$$x \oplus \perp = \perp$$

$$x \otimes \perp = \perp$$

$$x \ominus \perp = \perp$$

The Abstract Domain

- Our abstract domain forms a **lattice**
- A partial order $x \leq y \Leftrightarrow \gamma(x) \subseteq \gamma(y)$
- $x \leq y$ means that x is more precise than y
- $\top$ corresponds to all values in the concrete domain
    - (the least information)
- Every finite subset has a least upper bound (lub) &
  a greatest lower bound (glb)

$$
\begin{tikzcd}
 & \top \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
+ \arrow[dr, no head] & 0 \arrow[d, no head] & - \arrow[dl, no head] \\
 & \bot &
\end{tikzcd}
$$

Partial Orders

A relation $\preccurlyeq \ \subseteq \ D \times D$ on a set D is a **partial order** iff $\preccurlyeq$ is

1. Reflexive: $x \preccurlyeq x$
2. Anti-symmetric: $x \preccurlyeq y$ and $y \preccurlyeq x \Rightarrow x = y$
3. Transitive: $x \preccurlyeq y$ and $y \preccurlyeq z \Rightarrow x \preccurlyeq z$

- A set with a partial order is called a **poset**

Examples:

- If S is a set then $(2^S, \subseteq)$ is a poset
- $(\mathbb{Z}, \leq)$ is a poset

Hasse Diagram

- x immediate predecessor of y: if $x \preccurlyeq y$ and there is no z such that
$$x \preccurlyeq z \preccurlyeq y$$
- Hasse diagram: a directed acyclic graph where the vertices are elements of the set D
- There exists an edge $x \to y$ if x is an immediate predecessor of y

Example.

- $x \preccurlyeq y$, $y \preccurlyeq t$, $z \preccurlyeq t$, $x \preccurlyeq z$, $x \preccurlyeq t$
  $x \preccurlyeq x$, $y \preccurlyeq y$, $z \preccurlyeq z$, $t \preccurlyeq t$

$$\begin{tikzcd}
 & t & \\
y \arrow[ur] & & z \arrow[ul] \\
 & x \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Exercise

- $D_n = \{\text{all divisors of } n\},$ with $d \preccurlyeq d' \Leftrightarrow d \mid d'$
- Draw the Hasse diagram for $D_{12} = \{1, 2, 3, 4, 6, 12\}$

Exercise

- $D_n = \{\text{all divisors of } n\},$ with $d \preccurlyeq d' \Leftrightarrow d \mid d'$
- Draw the Hasse diagram for $D_{12} = \{1, 2, 3, 4, 6, 12\}$

$$\begin{tikzcd}
 & 12 & & \\
4 \arrow[ur] & & 6 \arrow[ul] & \\
 & 2 \arrow[ul] \arrow[ur] & & 3 \arrow[ul] \\
 & & 1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$D_{12} = \{1, 2, 3, 4, 6, 12\}$$

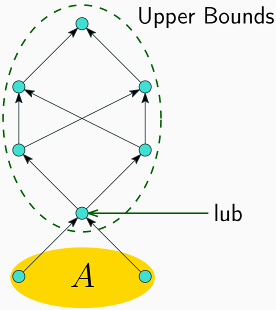
Total Order

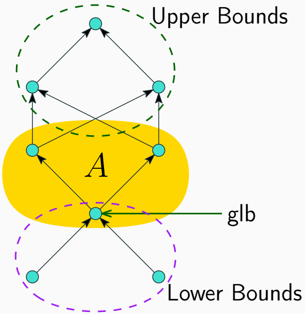
- Partial order: no guarantee that all elements can be compared to each other
- Total order (linear order): If for any two elements x and y at least one of $x \preccurlyeq y$ or $y \preccurlyeq x$ is true
- $(\mathbb{N}, \leq)$ is total order
- Hasse diagram is one-track

$$\begin{array}{c} \vdots \\ \uparrow \\ 4 \\ \uparrow \\ 3 \\ \uparrow \\ 2 \\ \uparrow \\ 1 \end{array}$$

Subset Bounds

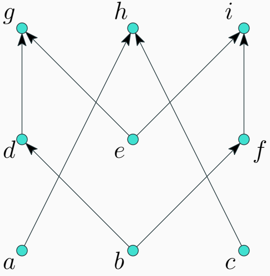
- Let $(X, \preccurlyeq)$ be a poset and let $A \subseteq X$ be any subset of X
- An element, $b \in X$, is a **lower bound** of A iff $b \preccurlyeq a$ for all $a \in A$
- An element, $m \in X$, is an **upper bound** of A iff $a \preccurlyeq m$ for all $a \in A$
- An element, $b \in X$, is the **greatest lower bound** (glb) of A iff the set of lower bounds of A is nonempty and if b is the greatest element of this set
- An element, $m \in X$, is the **least upper bound** (lub) of A iff the set of upper bounds of A is nonempty and if m is the least element of this set





Exercise

Find lower/upper bounds and glb/lub for these sets: $\{b, d\}$,$\{a, c\}$,$\{d, e, f\}$



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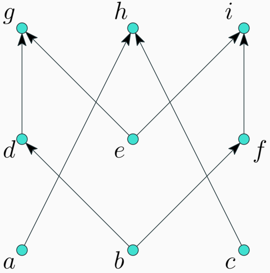

$\{\boldsymbol{b},\boldsymbol{d}\}$**:**

- Lower bounds: $\{b\}$ glb: b
- Upper bounds: $\{d,g\}$ lub: d because $d \preccurlyeq g$

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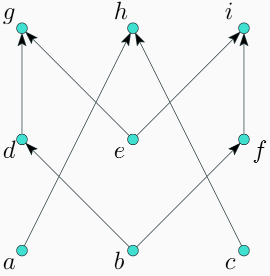
Find lower/upper bounds and glb/lub for these sets: $\{b, d\}$,$\{a, c\}$,$\{d, e, f\}$



$\{b, d\}$:

- Lower bounds: $\{b\}$ glb: b
- Upper bounds: $\{d, g\}$ lub: d because $d \preccurlyeq g$

$\{a, c\}$:

- Lower bounds: $\{\}$ no glb
- Upper bounds: $\{h\}$ lub: h

Exercise

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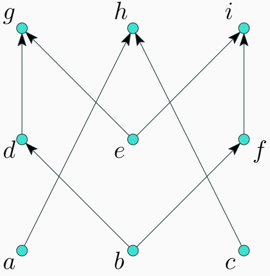

$\{\boldsymbol{b}, \boldsymbol{d}\}$:

- Lower bounds: $\{b\}$ glb: b
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$\{\boldsymbol{a}, \boldsymbol{c}\}$:

- Lower bounds: $\{\}$ no glb
- Upper bounds: $\{h\}$ lub: h

$\{\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}\}$:

- Lower bounds: $\{\}$ no glb
- Upper bounds: $\{\}$ no lub

Lattice

Poset $(D, \preccurlyeq)$ is called a lattice if

- For any $x, y \in D$, $\{x, y\}$ has a lub, which is denoted as $x \sqcup y$ (join)
- For any $x, y \in D$, $\{x, y\}$ has a glb, which is denoted as $x \sqcap y$ (meet)

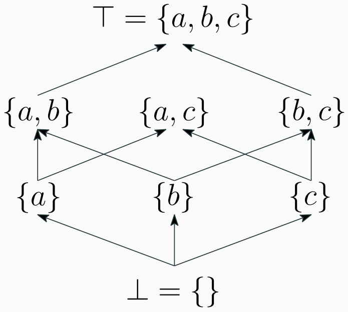
Example.

- For $(2^B, \subseteq)$: $x \sqcap y = x \cap y$, $x \sqcup y = x \cup y$
- For $(\mathbb{Z}, \leq)$: $x \sqcap y = \mathit{min}(x, y)$, $x \sqcup y = \mathit{max}(x, y)$

Complete Lattice

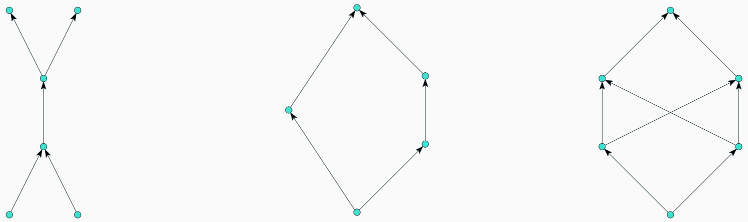
- **Complete lattice** is a poset in which any subset (finite or infinite) has a glb and a lub
  - Every finite lattice is complete
- A complete lattice must have:
  - a least element $\perp$
  - a greatest element $\top$

Example: Power Set Lattice



Exercise

- Which are the following posets are lattices?



- To show a poset is not a lattice, it suffices to find a pair that does not have an lub or a glb
- Two elements that don't have an lub or glb cannot be comparable
- View the upper/lower bounds on a pair as a sub-Hasse diagram:
  If there is no greatest/least element in this sub-diagram,
  then it is not a lattice

Exercise

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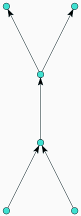
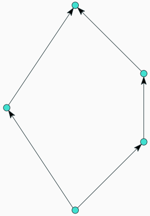
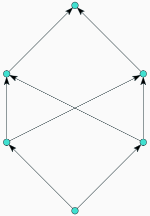


no





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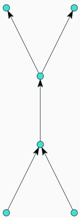
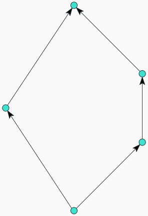
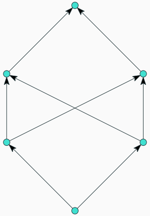


no



yes ✓



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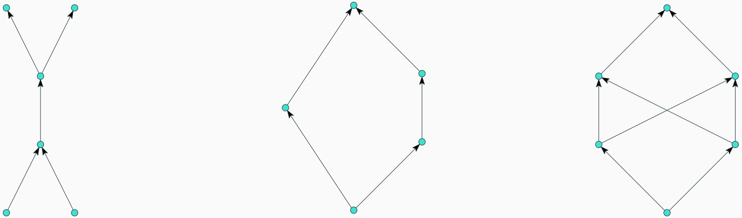

no yes ✓ no

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