CSCI 740 - Programming Language Theory

Lecture 23

Verification by Solving Horn Clauses

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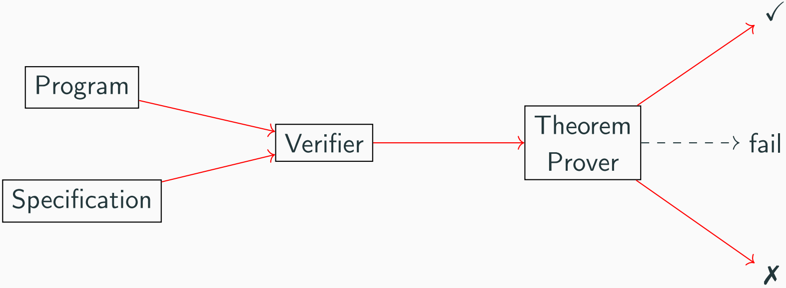
Recap: Automated Verification



Weakest Precondition Rules: Summary

c	$\mathsf{wp}(c, Q)$
$x := e$	$Q[x \mapsto e]$
$\mathsf{assume}(b)$	$b \to Q$
$\mathsf{assert}(b)$	$\mathsf{wp}(b \wedge Q)$
$\mathsf{havoc}(x)$	$\forall y. Q[x \mapsto y]$
$c_1; c_2$	$\mathsf{wp}(c_1, \mathsf{wp}(c_2, Q))$
$\texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2$	$b \to \mathsf{wp}(c_1, Q) \wedge \neg b \to \mathsf{wp}(c_2, Q)$
$\texttt{while}\ b\ \texttt{do}\ c$	$I \wedge \forall \vec{y}. \Big((I \wedge b \to \mathsf{wp}(c, I)) \wedge (I \wedge \neg b \to Q) \Big)[\vec{x} \mapsto \vec{y}]$ ($\vec{x}$ are variables modified in c and I is the loop invariant)

Loop Invariant

$$\frac{\vdash \{A \land b\}\ c\ \{A\}}{\vdash\ \{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \land \neg b\}}$$

$$\mathsf{wp}(\texttt{while}\ b\ \texttt{do}\ c,\ Q) = I \land \forall \vec{y}. \Big((I \land b \rightarrow \mathsf{wp}(c, I)) \land (I \land \neg b \rightarrow Q) \Big) [\vec{x} \mapsto \vec{y}]$$

($\vec{x}$ are variables modified in c and I is the loop invariant)

- Unfortunate reality: Inferring invariants automatically is undecideable
- This puts significant limits on the degree to which we can automate verification

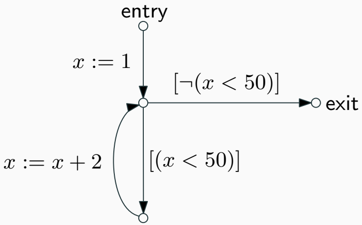
Loop Invariant

- Active research area:
  how to find loop invariants efficiently and automatically
- The simplest technique: guess-and-check!
- Given template of invariants (e.g., $? = ?$, $? \leq ?$), instantiate the holes with program variables and constants
- Check if it's an invariant; if not, try a different instantiation
- Abstract interpretation: popular approach to discover invariants
  - We will discuss it later in the course
- Today we discuss an alternative approach: reducing verification to checking the satisfiability of Horn clauses
- First, let's discuss control-flow graphs

Control Flow Graphs

- **Control Flow Graph (CFG):** graph representation of computation and control flow in the program
- Highlights the possible flow of execution

```
x = 1
while (x < 50) {
  x = x + 2
}
```



Control Flow Graphs

Control-Flow Graph (CFG) of a program P is a rooted directed graph $G = (V, E, \mathsf{entry}, \mathsf{exit})$ where

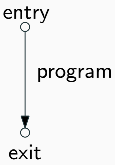
- $V \subseteq$ Label is a set of labels
- $E \subseteq$ Label $\times$ Action $\times$ Label is a set of arcs labeled by actions
- entry $\in V$ is the start state
- exit $\in V$ is the final state

Each action $f \in$ Action is a relation from program state to program state:

$$[\![f]\!] \subseteq \mathsf{state} \times \mathsf{state}$$

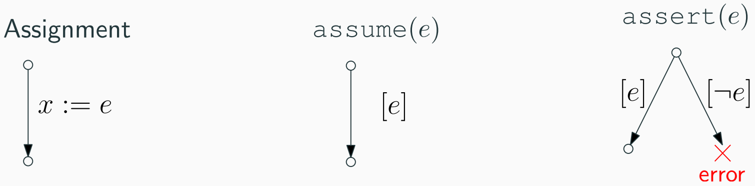
Generating Control-Flow Graphs

- Start with graph that has one **entry** and one **exit** node
- Draw an edge from entry to exit and label it with the entire program

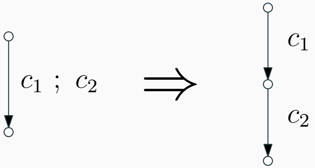


- Recursively decompose the program to have more edges with simpler labels
- When labels cannot be decomposed further, we are done

- Base cases

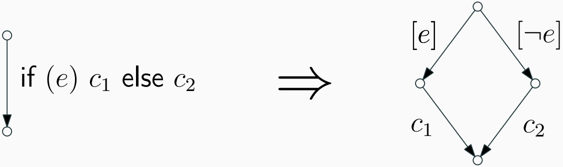


- Sequence of statements

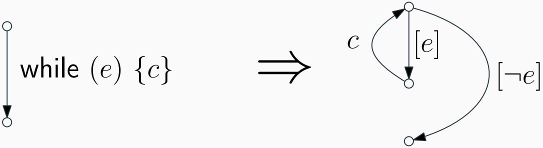


Control Structures

- Conditional statement



- While loop



Exercise: Convert to CFG

```
while(c2) {
  x = y - 1;
  y = z * 2;
  if (c1) x = y - z;
  z = 10;
}
z = x;
```

Exercise: Convert to CFG

```
while(c2) {
  x = y - 1;
  y = z * 2;
  if (c1) x = y - z;
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}
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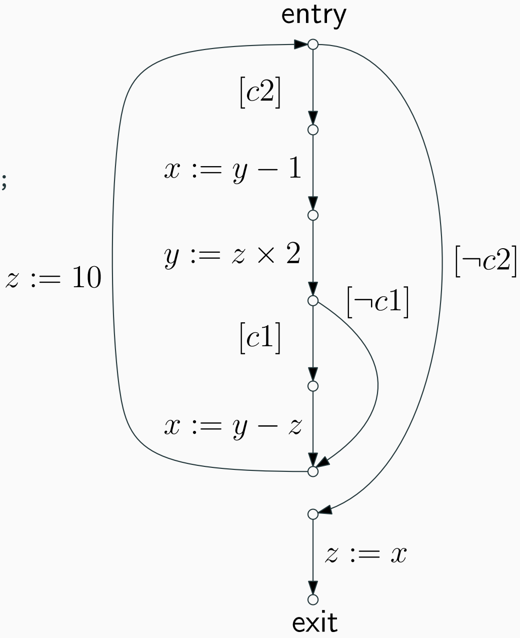

Example

- How to prove that the assertion does not fail in this program?

```
int n = 0;
while (true) {
  n = n + 1;
  assert(n≥−10);
  n = n − 1;
}
```

Example

- How to prove that the assertion does not fail in this program?

```
int n = 0;
while (true) {
  n = n + 1;
  assert(n≥−10);
  n = n − 1;
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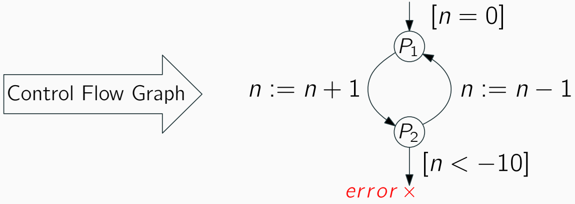

Example

- How to prove that the assertion does not fail in this program?

```
int n = 0;
while (true) {
  n = n + 1;
  assert(n≥−10);
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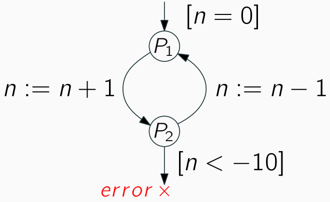
}
```

Control Flow Graph



- Let $P_i(n)$ denotes the reachable values of n in state P_i
- Initially we do not know the set of reachable values of n in each state
- P_i's can be any Boolean formula on n, for example:
  - $P_1(n) = (n \geq 0)$ and $P_2(n) = (n = -5) \vee (n > 0)$
  - $P_1(n) = (n = 0)$ and $P_2(n) =$ true (any value can reach it)
  - ...
- We can write constraints between P_i's according to CFG

Example

- How to prove that the assertion does not fail in this program?

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while (true) {
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  assert(n≥−10);
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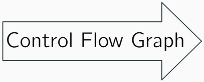
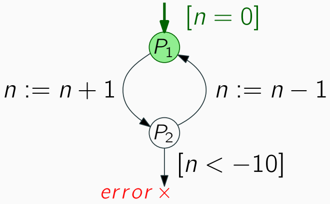
}
```





$$(n = 0) \quad \rightarrow \quad P_1(n)$$

Example

- How to prove that the assertion does not fail in this program?

```
int n = 0;
while (true) {
  n = n + 1;
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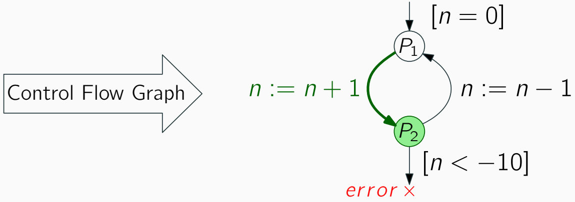


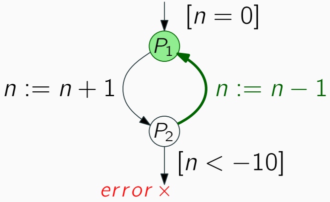
$$
\begin{aligned}
(n = 0) &\quad\rightarrow\quad P_1(n) \\
P_1(n) \wedge (n' = n + 1) &\quad\rightarrow\quad P_2(n')
\end{aligned}
$$

Example

- How to prove that the assertion does not fail in this program?

```
int n = 0;
while (true) {
  n = n + 1;
  assert(n≥−10);
  n = n − 1;
}
```

Control Flow Graph

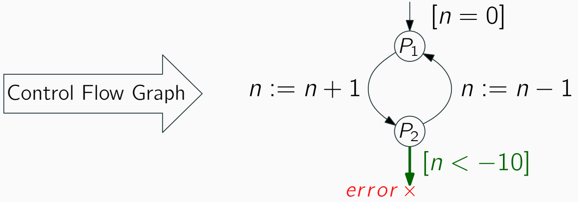


$$
\begin{aligned}
(n = 0) &\quad\to\quad P_1(n) \\
P_1(n) \wedge (n' = n + 1) &\quad\to\quad P_2(n') \\
P_2(n) \wedge (n' = n - 1) &\quad\to\quad P_1(n')
\end{aligned}
$$

Example

- How to prove that the assertion does not fail in this program?

```
int n = 0;
while (true) {
  n = n + 1;
  assert(n≥−10);
  n = n − 1;
}
```

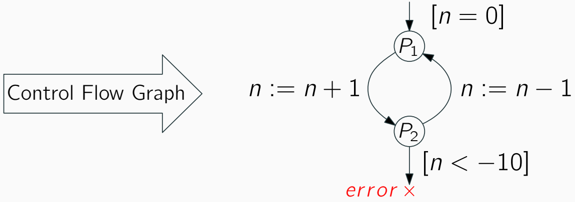


$$
\begin{aligned}
(n = 0) &\quad\rightarrow\quad P_1(n) \\
P_1(n) \wedge (n' = n + 1) &\quad\rightarrow\quad P_2(n') \\
P_2(n) \wedge (n' = n - 1) &\quad\rightarrow\quad P_1(n') \\
P_2(n) \wedge (n < -10) &\quad\rightarrow\quad \mathit{false}
\end{aligned}
$$

Example

- How to prove that the assertion does not fail in this program?

```
int n = 0;
while (true) {
  n = n + 1;
  assert(n≥−10);
  n = n − 1;
}
```



$$
\begin{aligned}
&\forall n. && (n = 0) && \rightarrow && P_1(n)\\
&\forall n, n'. && P_1(n) \wedge (n' = n + 1) && \rightarrow && P_2(n')\\
&\forall n, n'. && P_2(n) \wedge (n' = n - 1) && \rightarrow && P_1(n')\\
&\forall n. && P_2(n) \wedge (n < -10) && \rightarrow && \mathit{false}
\end{aligned}
$$

Example

- How to prove that the assertion does not fail in this program?

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int n = 0;
while (true) {
  n = n + 1;
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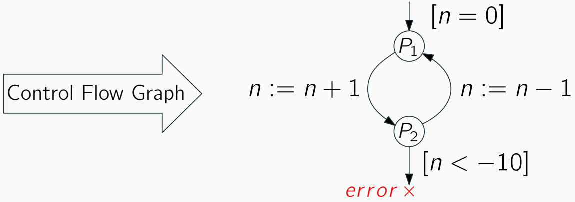

$$
\begin{array}{lll}
\forall n. & (n = 0) & \rightarrow \; P_1(n) \\
\forall n, n'. & P_1(n) \wedge (n' = n + 1) & \rightarrow \; P_2(n') \\
\forall n, n'. & P_2(n) \wedge (n' = n - 1) & \rightarrow \; P_1(n') \\
\forall n. & P_2(n) \wedge (n < -10) & \rightarrow \; \mathit{false}
\end{array}
$$

Solvable: $P_1(n) \equiv (n \geq 0)$ and $P_2(n) \equiv (n \geq 1)$

Demo

- Try `https://rise4fun.com/`

$$
\begin{array}{lll}
\forall n. & (n = 0) & \rightarrow \; P_1(n) \\
\forall n, n'. & P_1(n) \wedge (n' = n + 1) & \rightarrow \; P_2(n') \\
\forall n, n'. & P_2(n) \wedge (n' = n - 1) & \rightarrow \; P_1(n') \\
\forall n. & P_2(n) \wedge (n < -10) & \rightarrow \; \mathit{false}
\end{array}
$$

```
(set-logic HORN)
(declare-fun P1 (Int) Bool)
(declare-fun P2 (Int) Bool)
(assert (forall ((n Int)) (=> (= n 0) (P1 n) )))
(assert (forall ((n Int)(np Int)) (=> (and (P1 n) (= np (+ n 1)))
                                      (P2 np)))))
(assert (forall ((n Int)(np Int)) (=> (and (P2 n) (= np (- n 1)))
                                      (P1 np)))))
(assert (forall ((n Int)) (=> (and (P2 n) (< n (- 10))) false)))
(check-sat)
(get-model)
```

Horn Clause

- Horn clause is an implication of the form:

$$\forall \bar{v}.\ \underbrace{\Phi(\bar{v}) \land R_1(\bar{v}) \land \cdots \land R_n(\bar{v})}_{\text{body}} \longrightarrow \underbrace{R_0(\bar{v})}_{\text{head}}$$

- $\Phi(\bar{v})$ is an arithmetic formula (e.g. $x + 2y \leq z$)
- $R_i(\bar{v})$ is a relation symbol
- Head of the clause is either a relation symbol or $false$
- A solution for the Horn clause is an assignment of formulae to relation symbols for which the implication is valid

Verification by Solving Horn Clauses

Code

Safety Description

$$\forall \bar{v}.\ \Phi^0(\bar{v}) \wedge R_1^0(\bar{v}) \wedge \cdots \wedge R_n^0(\bar{v}) \rightarrow R_0^0(\bar{v})$$
$$\forall \bar{v}.\ \Phi^1(\bar{v}) \wedge R_1^1(\bar{v}) \wedge \cdots \wedge R_n^1(\bar{v}) \rightarrow R_0^1(\bar{v})$$
$$\vdots$$
$$\forall \bar{v}.\ \Phi^m(\bar{v}) \wedge R_1^m(\bar{v}) \wedge \cdots \wedge R_n^m(\bar{v}) \rightarrow R_0^m(\bar{v})$$
$$\forall \bar{v}.\ \Phi^i(\bar{v}) \wedge R_1^i(\bar{v}) \wedge \cdots \wedge R_n^i(\bar{v}) \rightarrow \mathit{false}$$

Horn Clause Solver

Exercise

- Convert to Horn clauses

```
int x,y;
assume(x≥0 ∧ y≥0);
while(x≠y) {
 if (x>y) then x:=x−y;
 else y:=y−x;
}
assert(x≠−1);
```

Exercise

- Convert to Horn clauses

```
int x,y;
assume(x≥0 ∧ y≥0);
while(x≠y) {
 if (x>y) then x:=x−y;
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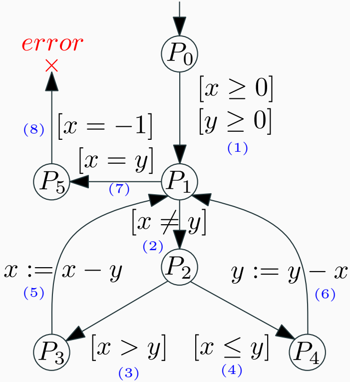

Exercise

- Convert to Horn clauses

$$\begin{array}{rccc}
 & \textit{true} & \rightarrow & P_0(x,y) \\
1) & P_0(x,y) \wedge (x \geq 0) \wedge (y \geq 0) & \rightarrow & P_1(x,y) \\
2) & P_1(x,y) \wedge (x \neq y) & \rightarrow & P_2(x,y) \\
3) & P_2(x,y) \wedge (x > y) & \rightarrow & P_3(x,y) \\
4) & P_2(x,y) \wedge (x \leq y) & \rightarrow & P_4(x,y) \\
5) & P_3(x,y) \wedge (x' = x - y) & \rightarrow & P_1(x',y) \\
6) & P_4(x,y) \wedge (y' = y - x) & \rightarrow & P_1(x,y') \\
7) & P_1(x,y) \wedge (x = y) & \rightarrow & P_5(x,y) \\
8) & P_5(x,y) \wedge (x = -1) & \rightarrow & \textit{false}
\end{array}$$