Programming Language Concepts

CSCI-344 **Recitation 10**
Term 20161

Type Inference

1 Introduction

You will undertake a few simple problems in order to develop basic skills with type inference.

2 Description

- (Adapted from Exercises 8 and 9 of Chapter 7 from *Programming Languages: Build, Prove, and Compare* (pp. 523 – 524).)
  - Find two constraints $C_1^{\sim}$ and $C_2^{\sim}$ and two substitutions θ_1 and θ_2 such that
    - $C_1^{\sim}$ has one or more free (unification) type variables,
    - $C_2^{\sim}$ has one or more free (unification) type variables,
    - $C_1^{\sim}$ is solved by θ_1,
    - $C_2^{\sim}$ is solved by θ_2,
    - $C_1^{\sim} \wedge C_2^{\sim}$ is solved by $\theta_2 \circ \theta_1$.
  - Find two constraints $C_1^{\sim}$ and $C_2^{\sim}$ and two substitutions θ_1 and θ_2 such that
    - $C_1^{\sim}$ has one or more free (unification) type variables,
    - $C_2^{\sim}$ has one or more free (unification) type variables,
    - $C_1^{\sim}$ is solved by θ_1,
    - $C_2^{\sim}$ is solved by θ_2,
    - $C_1^{\sim} \wedge C_2^{\sim}$ is *not* solved by *any* θ.
  - Find two constraints $C_1^{\sim}$ and $C_2^{\sim}$ and three substitutions θ_1, θ_2, and θ_3 such that
    - $C_1^{\sim}$ has one or more free (unification) type variables,
    - $C_2^{\sim}$ has one or more free (unification) type variables,
    - $C_1^{\sim}$ is solved by θ_1,
    - $C_2^{\sim}$ is solved by θ_2,
    - $C_1^{\sim} \wedge C_2^{\sim}$ is solved by θ_3,
    - $C_1^{\sim} \wedge C_2^{\sim}$ is *not* solved by $\theta_2 \circ \theta_1$.

- For each of the following `(val x e)` definitions, write down the constraints $C^{\sim}$ and type τ that are inferred for the expression `e` (use the derivation templates on the following pages), a most general substitution θ that solves the constraint, and the type scheme σ that is inferred for the variable `x` when type checked with the initial environment

$$\begin{aligned}\Gamma_0 = \{&\texttt{nil} \mapsto \forall\alpha\,.\,\alpha\ \mathsf{list}, \texttt{cons} \mapsto \forall\alpha\,.\,\alpha \times \alpha\ \mathsf{list} \to \alpha\ \mathsf{list},\\ &\texttt{null?} \mapsto \forall\alpha\,.\,\alpha\ \mathsf{list} \to \mathsf{bool}, \texttt{car} \mapsto \forall\alpha\,.\,\alpha\ \mathsf{list} \to \alpha, \texttt{cdr} \mapsto \forall\alpha\,.\,\alpha\ \mathsf{list} \to \alpha\ \mathsf{list},\\ &\texttt{+} \mapsto \forall\,.\,\mathsf{int} \times \mathsf{int} \to \mathsf{int}, \texttt{-} \mapsto \forall\,.\,\mathsf{int} \times \mathsf{int} \to \mathsf{int}, \texttt{*} \mapsto \forall\,.\,\mathsf{int} \times \mathsf{int} \to \mathsf{int}, \texttt{/} \mapsto \forall\,.\,\mathsf{int} \times \mathsf{int} \to \mathsf{int},\\ &\texttt{=} \mapsto \forall\,.\,\mathsf{int} \times \mathsf{int} \to \mathsf{bool}, \texttt{<} \mapsto \forall\,.\,\mathsf{int} \times \mathsf{int} \to \mathsf{bool}, \texttt{>} \mapsto \forall\,.\,\mathsf{int} \times \mathsf{int} \to \mathsf{bool}\}\end{aligned}$$

  - `(val singleton (lambda (x) (cons x nil)))`
  - `(val mcons (lambda (p x l) (if (p x) (cons x l) l)))`

3 Requirements and Submission

You may use the reference interpreter (see Appendix A), but there may only be one active laptop in each group.

At the end of class, submit the group's solutions as hard-copy; be sure to include the names of all group members in the submission.

A Interpreter

A reference nano-ML interpreter is available on the CS Department Linux systems (e.g., `glados.cs.rit.edu` and `queeg.cs.rit.edu` and ICLs 1 and 2) at:

`/usr/local/pub/mtf/plc/bin/nml`

Use the reference interpreter to check your code.

B Constraint-based Typing Rules

$\boxed{\sigma = \texttt{generalize}(\theta, A)}$

$$\forall \alpha_1, \ldots, \alpha_n.\ \tau[\mathrm{a}_1 \mapsto \alpha_1, \ldots, \mathrm{a}_n \mapsto \alpha_n] \quad \text{where } \{\mathrm{a}_1, \ldots, \mathrm{a}_n\} = \mathrm{futv}(\tau) - A$$

$\boxed{C^{\sim} \text{ solved by } \theta}$

$$\frac{}{\mathrm{T} \text{ solved by } \{\}} \qquad \frac{C_1^{\sim} \text{ solved by } \theta_1 \qquad \theta_1(C_2^{\sim}) \text{ solved by } \theta_2}{C_1^{\sim} \wedge C_2^{\sim} \text{ solved by } \theta_2 \circ \theta_1}$$

$$\frac{}{\mathsf{int} \sim \mathsf{int} \text{ solved by } \{\}} \qquad \frac{}{\mathsf{bool} \sim \mathsf{bool} \text{ solved by } \{\}}$$

$$\frac{\tau_1 \sim \tau_1' \wedge \cdots \tau_n \sim \tau_n' \wedge \tau_r \sim \tau_r' \text{ solved by } \theta}{\tau_1 \times \cdots \times \tau_n \to \tau_r \sim \tau_1' \times \cdots \times \tau_n' \to \tau_r' \text{ solved by } \theta} \qquad \frac{\tau \sim \tau' \text{ solved by } \theta}{\tau\ \mathsf{list} \sim \tau'\ \mathsf{list} \text{ solved by } \theta}$$

$$\frac{}{\mathrm{a} \sim \mathrm{a} \text{ solved by } \{\}} \qquad \frac{\mathrm{a} \notin \mathrm{futv}(\tau)}{\mathrm{a} \sim \tau \text{ solved by } \{\mathrm{a} \mapsto \tau\}} \qquad \frac{\mathrm{a} \notin \mathrm{futv}(\tau)}{\tau \sim \mathrm{a} \text{ solved by } \{\mathrm{a} \mapsto \tau\}}$$

$\boxed{C^{\sim}, \Gamma \vdash e : \tau}$

$$\frac{C_1^{\sim}, \Gamma \vdash e_1 : \tau_1 \qquad C_2^{\sim}, \Gamma \vdash e_2 : \tau_2 \qquad C_3^{\sim}, \Gamma \vdash e_3 : \tau_3}{C_1^{\sim} \wedge C_2^{\sim} \wedge C_3^{\sim} \wedge \tau_1 \sim \mathsf{bool} \wedge \tau_2 \sim \tau_3 \vdash \mathrm{IF}(e_1, e_2, e_3) : \tau_2} \ (\textsc{If})$$

$$\frac{\Gamma(x) = \forall \alpha_1, \ldots, \alpha_n \,.\, \tau \qquad \mathrm{a}_1, \ldots, \mathrm{a}_n \text{ fresh}}{\mathrm{T}, \Gamma \vdash \mathrm{VAR}(x) : \tau[\alpha_1 \mapsto \mathrm{a}_1, \ldots, \alpha_n \mapsto \mathrm{a}_n]} \ (\textsc{Var})$$

$$\frac{\mathrm{a}_1, \ldots, \mathrm{a}_n \text{ fresh} \qquad C^{\sim}, \Gamma\{x \mapsto \forall \,.\, \mathrm{a}_1, \ldots, x \mapsto \forall \,.\, \mathrm{a}_n\} \vdash e : \tau}{C^{\sim}, \Gamma \vdash \mathrm{LAMBDA}(\langle x_1, \ldots, x_n \rangle, e) : \mathrm{a}_1 \times \cdots \times \mathrm{a}_n \to \tau} \ (\textsc{Lambda})$$

$$\frac{C_f^{\sim}, \Gamma \vdash e_f : \tau_f \qquad C_1^{\sim}, \Gamma \vdash e_1 : \tau_1 \qquad \cdots \qquad C_n^{\sim}, \Gamma \vdash e_n : \tau_n \qquad \mathrm{a}_r \text{ fresh}}{C_f^{\sim} \wedge C_1^{\sim} \wedge \cdots \wedge C_n^{\sim} \wedge \tau_f \sim \tau_1 \times \cdots \tau_n \to \mathrm{a}_r, \Gamma \vdash \mathrm{APPLY}(e_f, e_1, \ldots, e_n) : \mathrm{a}_r} \ (\textsc{Apply})$$

$$\frac{\begin{array}{c} C_1^{\sim}, \Gamma \vdash e_1 : \tau_1 \qquad \cdots \qquad C_n^{\sim}, \Gamma \vdash e_n : \tau_n \\ C_1^{\sim} \wedge \cdots \wedge C_n^{\sim} \text{ solved by } \theta \qquad \theta \text{ idempotent} \\ C^{\sim} = \bigwedge\{\mathrm{a} \sim \theta(\mathrm{a}) \mid \mathrm{a} \in \mathrm{dom}(\theta) \cap \mathrm{futv}(\Gamma)\} \\ \sigma_1 = \texttt{generalize}(\theta(\tau_1), \mathrm{futv}(\Gamma) \cup \mathrm{futv}(C^{\sim})) \qquad \cdots \qquad \sigma_n = \texttt{generalize}(\theta(\tau_n), \mathrm{futv}(\Gamma) \cup \mathrm{futv}(C^{\sim})) \\ C_b^{\sim}, \Gamma\{x_1 \mapsto \sigma_1, \ldots, x_n \mapsto \sigma_n\} \vdash e_b : \tau_b \end{array}}{C^{\sim} \wedge C_b^{\sim}, \Gamma \vdash \mathrm{LET}(\langle x_1, e_1, \ldots, x_n, e_n \rangle, e_b) : \tau_b} \ (\textsc{Let})$$

$$\frac{\begin{array}{c} \mathrm{a}_1, \ldots, \mathrm{a}_n \text{ fresh} \\ C_1^{\sim}, \Gamma\{x_1 \mapsto \forall \,.\, \mathrm{a}_1, \ldots, x_n \mapsto \forall \,.\, \mathrm{a}_n\} \vdash e_1 : \tau_1 \qquad \cdots \qquad C_n^{\sim}, \Gamma\{x_1 \mapsto \forall \,.\, \mathrm{a}_1, \ldots, x_n \mapsto \forall \,.\, \mathrm{a}_n\} \vdash e_n : \tau_n \\ C_1^{\sim} \wedge \cdots \wedge C_n^{\sim} \wedge \mathrm{a}_1 \sim \tau_1 \wedge \cdots \wedge \mathrm{a}_n \sim \tau_n \text{ solved by } \theta \qquad \theta \text{ idempotent} \\ C^{\sim} = \bigwedge\{\mathrm{a} \sim \theta(\mathrm{a}) \mid \mathrm{a} \in \mathrm{dom}(\theta) \cap \mathrm{futv}(\Gamma)\} \\ \sigma_1 = \texttt{generalize}(\theta(\tau_1), \mathrm{futv}(\Gamma) \cup \mathrm{futv}(C^{\sim})) \qquad \cdots \qquad \sigma_n = \texttt{generalize}(\theta(\tau_n), \mathrm{futv}(\Gamma) \cup \mathrm{futv}(C^{\sim})) \\ C_b^{\sim}, \Gamma\{x_1 \mapsto \sigma_1, \ldots, x_n \mapsto \sigma_n\} \vdash e_b : \tau_b \end{array}}{C^{\sim} \wedge C_b^{\sim}, \Gamma \vdash \mathrm{LETREC}(\langle x_1, e_1, \ldots, x_n, e_n \rangle, e_b) : \tau_b} \ (\textsc{LetRec})$$

$\boxed{\langle d, \Gamma \rangle \to \Gamma'}$

$$\frac{\begin{array}{c} C^{\sim}, \Gamma \vdash e : \tau \qquad C^{\sim} \text{ solved by } \theta \\ \sigma = \texttt{generalize}(\theta(\tau), \emptyset) \end{array}}{\langle \mathrm{VAL}(\mathrm{x}, \mathrm{e}), \Gamma \rangle \to \Gamma\{x \mapsto \sigma\}} \ (\textsc{Val}) \qquad \frac{\begin{array}{c} \mathrm{a} \text{ fresh} \qquad C^{\sim}, \Gamma\{x \mapsto \forall \,.\, \mathrm{a}\} \vdash e : \tau \qquad C^{\sim} \wedge \mathrm{a} \sim \tau \text{ solved by } \theta \\ \sigma = \texttt{generalize}(\theta(\tau), \emptyset) \end{array}}{\langle \mathrm{VALREC}(\mathrm{x}, \mathrm{e}), \Gamma \rangle \to \Gamma\{x \mapsto \sigma\}} \ (\textsc{ValRec})$$

Derivation Templates

$$\mathrm{a}_x \text{ fresh} \qquad \Gamma_1 = \Gamma_0\{\mathtt{x} \mapsto \forall\,.\,\mathrm{a}_x\}$$

$$\_\_ \text{ fresh}$$

$$\text{(Var)}\ \dfrac{\Gamma_1(\mathtt{cons}) = \forall\alpha\,.\,\alpha \times \alpha\ \mathsf{list} \to \alpha\ \mathsf{list} \qquad \_\_ \text{ fresh}}{\_\_, \Gamma_1 \vdash \mathtt{cons} : \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$\text{(Var)}\ \dfrac{\Gamma_1(\mathtt{x}) = \forall\,.\,\mathrm{a}_x}{\_\_, \Gamma_1 \vdash \mathtt{x} : \_\_}$$

$$\text{(Var)}\ \dfrac{\Gamma_1(\mathtt{nil}) = \forall\alpha\,.\,\alpha\ \mathsf{list} \qquad \_\_ \text{ fresh}}{\_\_, \Gamma_1 \vdash \mathtt{nil} : \_\_\_\_\_}$$

$$\text{(Apply)}\ \dfrac{\cdots}{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_, \Gamma_1 \vdash \mathtt{(cons\ x\ nil)} : \_\_}$$

$$\text{(Lambda)}\ \dfrac{\cdots}{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_, \Gamma_0 \vdash \mathtt{(lambda\ (x)\ (cons\ x\ nil))} : \_\_\_\_\_\_\_}$$

a_p, a_x, a_l fresh $\qquad \Gamma_1 = \Gamma_0\{\mathtt{p} \mapsto \forall . a_p, \mathtt{x} \mapsto \forall . a_x, \mathtt{l} \mapsto \forall . a_l\}$

__ fresh

(VAR)
$$\frac{\Gamma_1(\mathtt{p}) = \forall . a_p}{\_, \Gamma_1 \vdash \mathtt{p} : \_\_}$$

(VAR)
$$\frac{\Gamma_1(\mathtt{x}) = \forall . a_x}{\_, \Gamma_1 \vdash \mathtt{x} : \_\_}$$

(APPLY)
$$\underline{\qquad\qquad\qquad\qquad}, \Gamma_1 \vdash \mathtt{(p\ x)} : \_\_$$

__ fresh

(VAR)
$$\frac{\Gamma_1(\mathtt{cons}) = \forall \alpha . \alpha \times \alpha \text{ list} \to \alpha \text{ list} \qquad \_\_ \text{ fresh}}{\_, \Gamma_1 \vdash \mathtt{cons} : \underline{\qquad\qquad\qquad\qquad}}$$

(VAR)
$$\frac{\Gamma_1(\mathtt{x}) = \forall . a_x}{\_, \Gamma_1 \vdash \mathtt{x} : \_\_}$$

(VAR)
$$\frac{\Gamma_1(\mathtt{l}) = \forall . a_l}{\_, \Gamma_1 \vdash \mathtt{l} : \_\_}$$

(APPLY)
$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}, \Gamma_1 \vdash \mathtt{(cons\ x\ l)} : \_\_$$

(VAR)
$$\frac{\Gamma_1(\mathtt{l}) = \forall . a_l}{\_, \Gamma_1 \vdash \mathtt{l} : \_\_}$$

(IF)
$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad},$$
$$\Gamma_1 \vdash \mathtt{(if\ (p\ x)\ (cons\ x\ l)\ l)} : \_\_$$

(LAMBDA)
$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad},$$
$$\Gamma_0 \vdash \mathtt{(lambda\ (p\ x\ l)\ (if\ (p\ x)\ (cons\ x\ l)\ l))} : \underline{\qquad\qquad\qquad\qquad}$$

Solutions

- satisfying/solving conjunctions
  - Consider $C_1^\sim = \mathrm{a} \sim \mathsf{int}$, $C_2^\sim = \mathrm{b} \sim \mathsf{bool}$, $\theta_1 = \{\mathrm{a} \mapsto \mathsf{int}\}$, and $\theta_2 = \{\mathrm{b} \mapsto \mathsf{bool}\}$.
    - $\mathrm{futv}(C_1^\sim) = \{\mathrm{a}\}$,
    - $\mathrm{futv}(C_2^\sim) = \{\mathrm{b}\}$,
    - $\theta_1(C_1^\sim) = \mathsf{int} \sim \mathsf{int}$ is satisfied (so $C_1^\sim$ is solved by θ_1),
    - $\theta_2(C_2^\sim) = \mathsf{bool} \sim \mathsf{bool}$ is satisfied (so $C_2^\sim$ is solved by θ_2),
    - $(\theta_2 \circ \theta_1)(C_1^\sim \wedge C_2^\sim) = \mathsf{int} \sim \mathsf{int} \wedge \mathsf{bool} \sim \mathsf{bool}$ is satisfied (so $C_1^\sim \wedge C_2^\sim$ is solved by $\theta_2 \circ \theta_1$),
  - Consider $C_1^\sim = \mathrm{a} \sim \mathsf{int}$, $C_2^\sim = \mathrm{a} \sim \mathsf{bool}$, $\theta_1 = \{\mathrm{a} \mapsto \mathsf{int}\}$, and $\theta_2 = \{\mathrm{a} \mapsto \mathsf{bool}\}$.
    - $\mathrm{futv}(C_1^\sim) = \{\mathrm{a}\}$,
    - $\mathrm{futv}(C_2^\sim) = \{\mathrm{b}\}$,
    - $\theta_1(C_1^\sim) = \mathsf{int} \sim \mathsf{int}$ is satisfied (so $C_1^\sim$ is solved by θ_1),
    - $\theta_2(C_2^\sim) = \mathsf{bool} \sim \mathsf{bool}$ is satisfied (so $C_2^\sim$ is solved by θ_2),
    - $\theta(C_1^\sim \wedge C_2^\sim) = \theta(\mathrm{a}) \sim \mathsf{int} \wedge \theta(\mathrm{b}) \sim \mathsf{bool}$ is *not* satisfied for *any* θ (so $C_1^\sim \wedge C_2^\sim$ is *not* solved by *any* θ).
  - Consider $C_1^\sim = \mathrm{a} \sim \mathsf{int}$, $C_2^\sim = \mathrm{a} \sim \mathrm{b}$, $\theta_1 = \{\mathrm{a} \mapsto \mathsf{int}\}$, $\theta_2 = \{\mathrm{a} \mapsto \mathrm{b}\}$, $\theta_3 = \{\mathrm{a} \mapsto \mathsf{int}, \mathrm{b} \mapsto \mathsf{int}\}$.
    - $\mathrm{futv}(C_1^\sim) = \{\mathrm{a}\}$,
    - $\mathrm{futv}(C_2^\sim) = \{\mathrm{a}, \mathrm{b}\}$,
    - $\theta_1(C_1^\sim) = \mathsf{int} \sim \mathsf{int}$ is satisfied (so $C_1^\sim$ is solved by θ_1),
    - $\theta_2(C_2^\sim) = \mathrm{b} \sim \mathrm{b}$ is satisfied (so $C_2^\sim$ is solved by θ_2),
    - $\theta_3(C_1^\sim \wedge C_2^\sim) = \mathsf{int} \sim \mathsf{int} \wedge \mathsf{int} \sim \mathsf{int}$ is satisfied (so $C_1^\sim \wedge C_2^\sim$ is solved by θ_3),
    - $(\theta_2 \circ \theta_1)(C_1^\sim \wedge C_2^\sim) = \mathsf{int} \sim \mathsf{int} \wedge \mathsf{int} \sim \mathrm{b}$ is *not* satisfied (so $C_1^\sim \wedge C_2^\sim$ is *not* solved by $\theta_2 \circ \theta_1$).
- type inference examples
  - `(val singleton (lambda (x) (cons x nil)))`
    - $C^\sim = \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{a}_1' \times \mathrm{a}_1'\ \mathsf{list} \to \mathrm{a}_1'\ \mathsf{list} \sim \mathrm{a}_x \times \mathrm{a}_2'\ \mathsf{list} \to \mathrm{a}_r$
    - $\tau = \mathrm{a}_x \to \mathrm{a}_r$
    - $\theta = \{\mathrm{a}_1' \mapsto \mathrm{a}_x, \mathrm{a}_2' \mapsto \mathrm{a}_x, \mathrm{a}_r \mapsto \mathrm{a}_x\ \mathsf{list}\}$
    - $\sigma = \forall \alpha_x \,.\, \alpha_x \to \alpha_x\ \mathsf{list}$
  - `(val mcons (lambda (p x l) (if (p x) (cons x l) l)))`
    - $C^\sim = \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{a}_p \sim \mathrm{a}_x \to \mathrm{a}_{r1} \wedge \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{a}_1' \times \mathrm{a}_1'\ \mathsf{list} \to \mathrm{a}_1'\ \mathsf{list} \sim \mathrm{a}_x \times \mathrm{a}_l \to \mathrm{a}_{r2} \wedge \mathrm{T} \wedge \mathrm{a}_{r1} \sim \mathsf{bool} \wedge \mathrm{a}_{r2} \sim \mathrm{a}_l$
    - $\tau = \mathrm{a}_p \times \mathrm{a}_x \times \mathrm{a}_l \to \mathrm{a}_{r2}$
    - $\theta = \{\mathrm{a}_p \mapsto \mathrm{a}_x \to \mathsf{bool}, \mathrm{a}_l \mapsto \mathrm{a}_x\ \mathsf{list}, \mathrm{a}_1' \mapsto \mathrm{a}_x, \mathrm{a}_{r1} \mapsto \mathsf{bool}, \mathrm{a}_{r2} \mapsto \mathrm{a}_x\ \mathsf{list}\}$
    - $\sigma = \forall \alpha_x \,.\, (\alpha_x \to \mathsf{bool}) \times \alpha_x \times \alpha_x\ \mathsf{list} \to \alpha_x\ \mathsf{list}$

Derivation Templates

$$\mathrm{a}_x \text{ fresh} \qquad \Gamma_1 = \Gamma_0\{\mathtt{x} \mapsto \forall\,.\,\mathrm{a}_x\}$$

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$$\mathrm{a}_r \text{ fresh}$$

$$\text{(VAR)}\ \frac{\Gamma_1(\mathtt{cons}) = \forall\alpha\,.\,\alpha \times \alpha \ \mathsf{list} \to \alpha\ \mathsf{list} \qquad \mathrm{a}'_1 \text{ fresh}}{\mathrm{T}, \Gamma_1 \vdash \mathtt{cons} : \mathrm{a}'_1 \times \mathrm{a}'_1\ \mathsf{list} \to \mathrm{a}'_1\ \mathsf{list}}$$

$$\text{(VAR)}\ \frac{\Gamma_1(\mathtt{x}) = \forall\,.\,\mathrm{a}_x}{\mathrm{T}, \Gamma_1 \vdash \mathtt{x} : \mathrm{a}_x}$$

$$\text{(VAR)}\ \frac{\Gamma_1(\mathtt{nil}) = \forall\alpha\,.\,\alpha\ \mathsf{list} \qquad \mathrm{a}'_2 \text{ fresh}}{\mathrm{T}, \Gamma_1 \vdash \mathtt{nil} : \mathrm{a}'_2\ \mathsf{list}}$$

$$\text{(APPLY)}\ \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{a}'_1 \times \mathrm{a}'_1\ \mathsf{list} \to \mathrm{a}'_1\ \mathsf{list} \sim \mathrm{a}_x \times \mathrm{a}'_2\ \mathsf{list} \to \mathrm{a}_r, \Gamma_1 \vdash \mathtt{(cons\ x\ nil)} : \mathrm{a}_r$$

$$\text{(LAMBDA)}\ \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{a}'_1 \times \mathrm{a}'_1\ \mathsf{list} \to \mathrm{a}'_1\ \mathsf{list} \sim \mathrm{a}_x \times \mathrm{a}'_2\ \mathsf{list} \to \mathrm{a}_r, \Gamma_0 \vdash \mathtt{(lambda\ (x)\ (cons\ x\ nil))} : \mathrm{a}_x \to \mathrm{a}_r$$

$$a_p, a_x, a_l \text{ fresh} \qquad \Gamma_1 = \Gamma_0\{\texttt{p} \mapsto \forall\,.\,a_p, \texttt{x} \mapsto \forall\,.\,a_x, \texttt{l} \mapsto \forall\,.\,a_l\}$$

a_{r1} fresh

$$\dfrac{\dfrac{\Gamma_1(\texttt{p}) = \forall\,.\,a_p}{T, \Gamma_1 \vdash \texttt{p} : a_p}\,(\text{VAR}) \qquad \dfrac{\Gamma_1(\texttt{x}) = \forall\,.\,a_x}{T, \Gamma_1 \vdash \texttt{x} : a_x}\,(\text{VAR})}{T \wedge T \wedge a_p \sim a_x \to a_{r1}, \Gamma_1 \vdash \texttt{(p x)} : a_{r1}}\,(\text{APPLY})$$

a_{r2} fresh

$$\dfrac{\dfrac{\Gamma_1(\texttt{cons}) = \forall\alpha\,.\,\alpha \times \alpha \ \mathsf{list} \to \alpha \ \mathsf{list} \qquad a'_1 \text{ fresh}}{T, \Gamma_1 \vdash \texttt{cons} : a'_1 \times a'_1 \ \mathsf{list} \to a'_1 \ \mathsf{list}}\,(\text{VAR}) \qquad \dfrac{\Gamma_1(\texttt{x}) = \forall\,.\,a_x}{T, \Gamma_1 \vdash \texttt{x} : a_x}\,(\text{VAR}) \qquad \dfrac{\Gamma_1(\texttt{l}) = \forall\,.\,a_l}{T, \Gamma_1 \vdash \texttt{l} : a_l}\,(\text{VAR})}{T \wedge T \wedge T \wedge a'_1 \times a'_1 \ \mathsf{list} \to a'_1 \ \mathsf{list} \sim a_x \times a_l \to a_{r2}, \Gamma_1 \vdash \texttt{(cons x l)} : a_{r2}}\,(\text{APPLY})$$

$$\dfrac{\Gamma_1(\texttt{l}) = \forall\,.\,a_l}{T, \Gamma_1 \vdash \texttt{l} : a_l}\,(\text{VAR})$$

$$\dfrac{\dfrac{\ldots}{\begin{array}{l} T \wedge T \wedge a_p \sim a_x \to a_{r1} \wedge T \wedge T \wedge T \wedge a'_1 \times a'_1 \ \mathsf{list} \to a'_1 \ \mathsf{list} \sim a_x \times a_l \to a_{r2} \wedge T \wedge a_{r1} \sim \mathsf{bool} \wedge a_{r2} \sim a_l, \\ \Gamma_1 \vdash \texttt{(if (p x) (cons x l) l)} : a_{r2} \end{array}}\,(\text{IF})}{\begin{array}{l} T \wedge T \wedge a_p \sim a_x \to a_{r1} \wedge T \wedge T \wedge T \wedge a'_1 \times a'_1 \ \mathsf{list} \to a'_1 \ \mathsf{list} \sim a_x \times a_l \to a_{r2} \wedge T \wedge a_{r1} \sim \mathsf{bool} \wedge a_{r2} \sim a_l, \\ \Gamma_0 \vdash \texttt{(lambda (p x l) (if (p x) (cons x l) l))} : a_p \times a_x \times a_l \to a_{r2} \end{array}}\,(\text{LAMBDA})$$